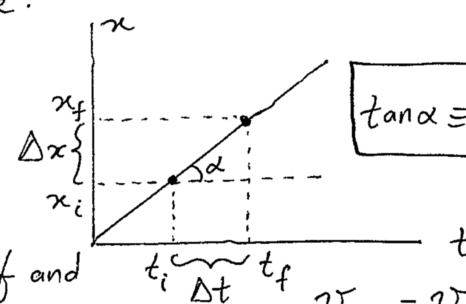
2.1 Uniform Motion

An object's motion is uniform if and only if its $x-t$ graph is a straight line.

Slope is $\boxed{v_{avg} \equiv \dfrac{\Delta x}{\Delta t}}$

$[v_{avg}] = \left\{ \dfrac{km}{hr}, \ mph \right\}$



$\boxed{\tan\alpha \equiv \dfrac{\Delta x}{\Delta t}}$ is slope

The motion is uniform if and only if v_x is constant.

$v_{avg} = v_x$ since $\dfrac{\Delta x}{\Delta t}$ is constant

1) Steeper slopes correspond to faster speeds
2) Sign of the slope determines direction of motion

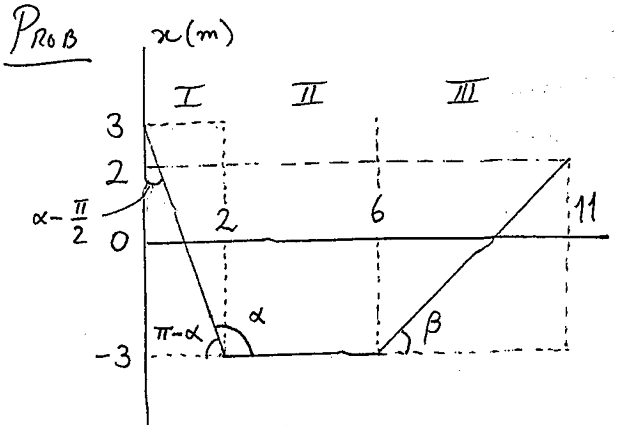
Speed is magnitude, irrespective of direction, $v = |v_x|$.

Velocity v_x is slope, has directionality in it as its sign.

$$\underbrace{v_x = \frac{x_f - x_i}{t_f - t_i}} = \frac{\Delta x}{\Delta t} \Rightarrow v_x = \frac{x_f - x_i}{\Delta t}$$

$\boxed{x_f = x_i + v_x \Delta t}$ uniform motion w/ constant v_x

PROB

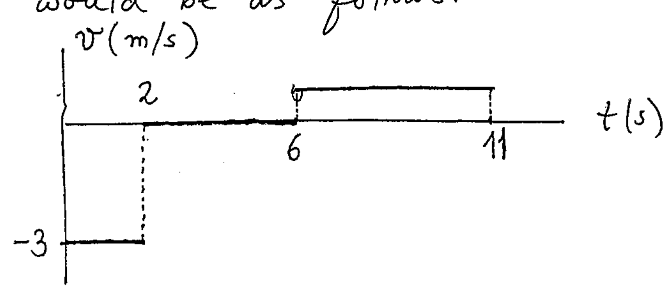


$\tan\alpha = v_I = \dfrac{-6}{2} = -3 \ m/s$

$v_{II} = 0 \ m/s$

$v_{III} = \tan\beta = \dfrac{5}{5} = 1 \ m/s$

The velocity-time graph would be as follows.



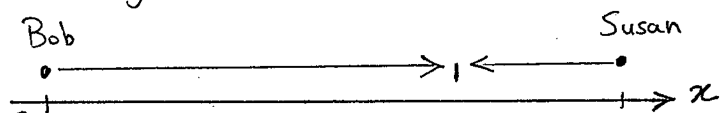
$$\tan(\pi - \alpha) = \frac{\sin(\pi-\alpha)}{\cos(\pi-\alpha)}$$

$$= \frac{\sin\pi \cos\alpha - \cos\pi \sin\alpha}{\cos\pi\cos\alpha + \sin\pi\sin\alpha}$$

$$= -\tan\alpha$$

$$\tan\alpha = -\tan(\pi - \alpha) = -\frac{6m}{2s} = -3 \ m/s$$

PROB. Bob leaves home in Chicago at 9:00 am and travels east at a steady 60 mph. Susan, 400 miles to the east in Pittsburgh, leaves at the same time and travels west at a steady 40 mph. Where will they meet?



Origin

$x_{0,B} = 0$ miles	$x_{1,B}$	$x_{0,s} = 400$ miles
$v_{x,B} = 60$ mph	$x_{1,s}$	$v_{x,s} = -40$ mph
$t_0 = 0s$	@ t_1	$t_0 = 0s$

Uniform motion $\rightarrow$ $x_f = x_i + v_x \Delta t$

They meet when $x_{1,B} = x_{1,s}$ at the same time, t_1.

$$\begin{cases} x_{1,B} = x_{0,B} + v_{x,B}(t_1 - t_0) \\ x_{1,s} = x_{0,s} + v_{x,s}(t_1 - t_0) \end{cases}$$

$$\left.\begin{cases} x_{1,B} = 60t_1 \\ x_{1,s} = 400 - 40t_1 \end{cases}\right\} \quad x_{1,B} = x_{1,s} \rightarrow 60t_1 = 400 - 40t_1$$

$$100t_1 = 400$$

$$t_1 = 4 \text{ hrs.}$$

$\rightarrow$ Then, $x_{1,B} = 60t_1 = 240$ miles.

$\rightarrow$ Does this number makes sense?

✓ – It's more than half-way from Chicago.

✓ – The answer is between 0-400 miles.

2.2 Instantaneous Velocity

$$\boxed{v_x \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}}$$

The instantaneous velocity at time t is the average velocity during a time interval Δt, centered on t, as Δt approaches 0.

The instantaneous velocity is the slope of the line that is tangent to the x-t curve at time t.

<u>Derrivatives:</u> If $x = 2t^2$ m, calculate v_x.

$$v_x = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}, \quad \Delta x = x_{t+\Delta t} - x_t$$

$$\Delta x = 2(t+\Delta t)^2 - 2t^2 = 4t\Delta t + 2(\Delta t)^2$$

$$v_x = \lim_{\Delta t \to 0} \frac{4t\Delta t + 2(\Delta t)^2}{\Delta t}$$

$$v_x = \lim_{\Delta t \to 0} (4t + 2\Delta t) = 4t \text{ m/s}$$

For polynomials, by mathematical induction, this leads to certain rules for differentiation,

$$u = ct^n \rightarrow \boxed{\frac{du}{dt} = cnt^{n-1}}, \quad u = 2t^2 \rightarrow \frac{du}{dt} = 4t$$

$$x = 5/t^2 = 5t^{-2} \rightarrow \frac{dx}{dt} = -10t^{-3} = -10/t^3$$

If $\boxed{x = \text{Constant} \rightarrow dx/dt = 0.}$

Associative $\boxed{\frac{d}{dt}(u+w) = \frac{du}{dt} + \frac{dw}{dt}}$

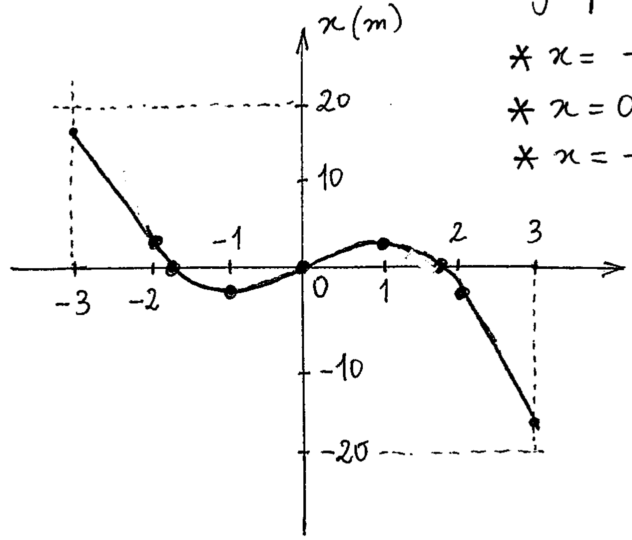
PROB. $x = (-t^3 + 3t)$ m

a) $x(t=2s) = ?$ $\quad x(t=2s) = -8 + 6 = -2$ m

$$v_x = \frac{dx}{dt} = (-3t^2 + 3) \frac{m}{s}$$

Velocity @ t = 2s → $v_x(t=2s) = -12 + 3 = -9 \frac{m}{s}$

b) Draw $x-t$ and v_x-t graphs during the interval $-3s \leq t \leq 3s$.



$* \; x = -t^3+3t = t(-t^2+3)$

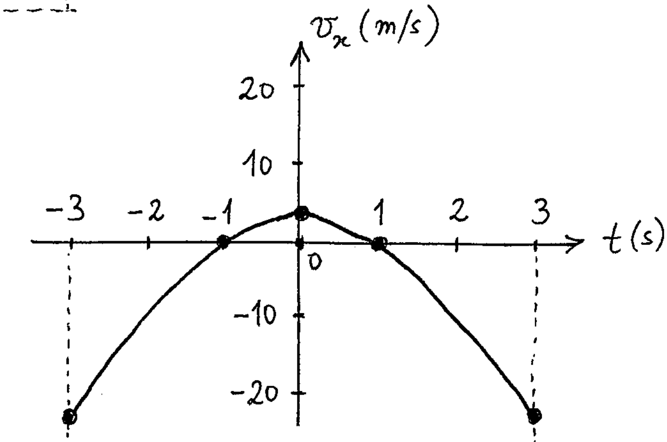
$* \; x = 0m$ at $t=0$ and $t = \pm\sqrt{3} \sim \pm 1.7$

$* \; x = -2\,m$ at $t=2s$ and $x=+2m$ at $t=-2s$

$* \; \frac{dx}{dt} = -3t^2+3$ vanishes at $t = \pm 1s$.

$*$ At $t=3s$, $x = -18\,m$

$*$ At $t=-3s$, $x = +18\,m$



$v_x = -3t^2+3 \; \frac{m}{s}$

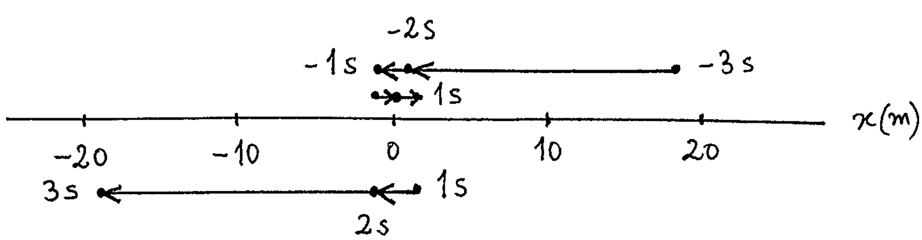
$*$ At $t=0s$, $v_x = 3\,m/s$

$* \; \frac{dv_x}{dt} = 0$ at $t=0\,s$

$* \; v_x = 0$ at $t = \pm 1\,s$

$*$ At $t=\pm 3s$, $v_x = -24 \; \frac{m}{s}$

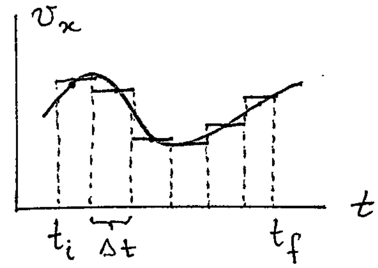
c) Draw the motion diagram



2.3 Finding Position from Velocity

For constant velocity $x_f = x_i + v_x \Delta t$

<u>When v_x is not constant:</u> Approximate the velocity curve as a series of constant velocity steps of width Δt.



$$\Delta x = \sum_{k=1}^{N} v_{x,k} \Delta t$$

$$x_f \approx x_i + \sum_{k=1}^{N} v_{x,k} \Delta t \leftarrow \text{Approximate}$$

As $\Delta t \to 0$, $N \to \infty$,

$$x_f = x_i + \lim_{\Delta t \to 0} \sum_{k=0}^{N} v_{x,k} \Delta t = x_i + \int_{t_i}^{t_f} v_x \, dt$$

Geometrical Interpretation of integral: Area under v_x curve.

$$x_f = x_i + [\text{Area under the } v_x \text{ curve between } t_i \text{ and } t_f]$$

Unit of the area under v_x curve : $[v_x][t] = \frac{m}{s} s = m$.

<u>Integrals:</u> Evaluating an integral is equivalent to finding the area under a graph/function.

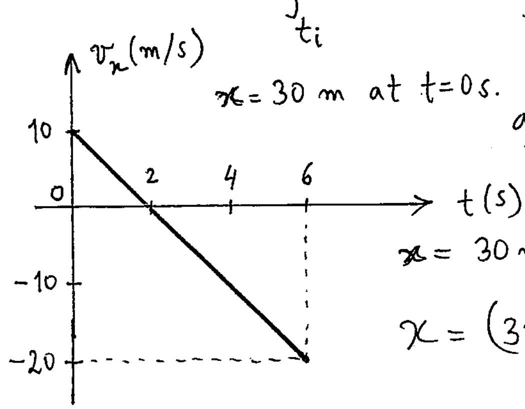
Definite integral : Boundaries are defined

$$\int_{t_i}^{t_f} u \, dt = \int_{t_i}^{t_f} c t^n \, dt = \left. \frac{c t^{n+1}}{n+1} \right|_{t_i}^{t_f} = \frac{c t_f^{n+1}}{n+1} - \frac{c t_i^{n+1}}{n+1}$$

$$= \frac{c}{n+1} \left(t_f^{n+1} - t_i^{n+1} \right), \quad n \neq 1$$

Associative : $\displaystyle\int_{t_i}^{t_f} (u+w) \, dt = \int_{t_i}^{t_f} u \, dt + \int_{t_i}^{t_f} w \, dt$

PROB.



$x = 30$ m at $t = 0$ s.

a) Where is the particle's turning point?

$v_x = At + B$, $B = 10 \frac{m}{s}$, $A = -5 \frac{m}{s^2}$

$$x = 30 \text{ m} + \int_0^t (-5t + 10) \, dt \ \frac{m}{s} = 30 + \left. \left(-\frac{5}{2} t^2 + 10t \right) \right|_0^t$$

$x = \left(30 + 10t - \frac{5}{2} t^2\right)$ m $\rightarrow$ It is a <u>parabola</u>

The turning point is when $v_x = 0$, i.e. at $t = 2s$.

$x(t=2s) = 30 + (10 \times 2) - \frac{5}{2} 4 = 40 \text{ m}$

b) At what time the particle reaches the origin ($x = 0 m$)?

$x = 30 + 10t - \frac{5}{2} t^2 = 0$, $\Delta = 100 + 300 = 400$

$t_{1,2} = \frac{-10 \pm \sqrt{400}}{-5} = \begin{cases} -2 \text{ s} \times \\ 6 \text{ s} \checkmark \end{cases}$ → Refers to a time before the problem began.

2.4 Motion with Constant Acceleration

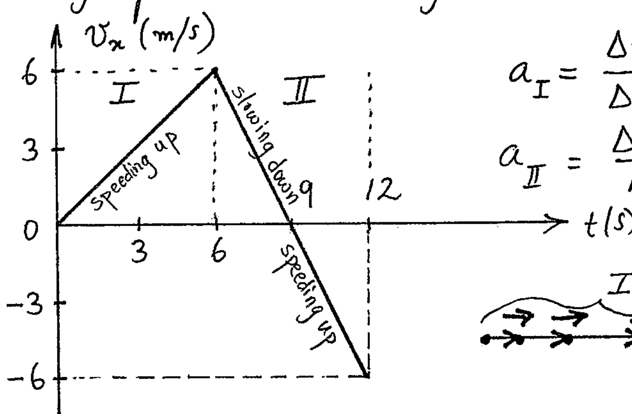
Acceleration is rate of change of velocity.

Average acceleration $\boxed{a_{avg} = \frac{\Delta v}{\Delta t}}$ $[(m/s)/s \equiv m/s^2]$

Uniformly accelerated motion: If and only if a_x is constant and not changing. The objects velocity vs. time graph is a straight line and a_x is its slope. $\boxed{a_{avg} = a_x}$

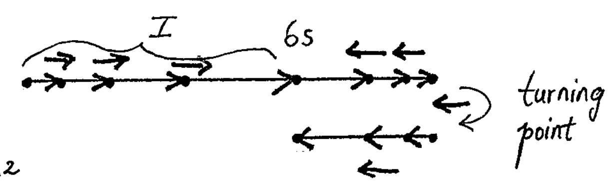
Positive + Negative values of a_x does not correspond to "speeding up" or "slowing down".

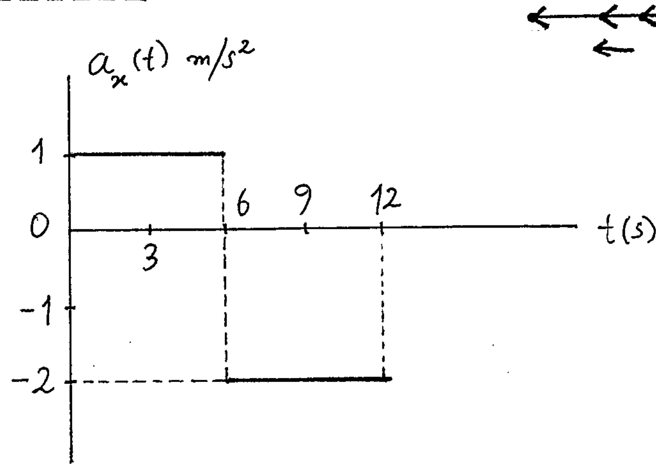
PROB.



$a_I = \frac{\Delta v_x}{\Delta t} = \frac{6 \text{ m/s} - 0 \text{ m/s}}{6s - 0s} = 1 \frac{m}{s^2}$

$a_{II} = \frac{\Delta v_x}{\Delta t} = \frac{-6 \text{ m/s} - 6 \text{ m/s}}{12s - 9s} = -2 \frac{m}{s^2}$

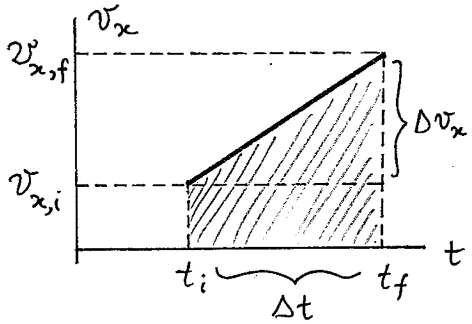




Kinematic Equations for Constant Acceleration

$$a_x = \frac{\Delta v_x}{\Delta t} = \frac{v_{x,f} - v_{x,i}}{\Delta t}$$

$$\boxed{v_{x,f} = v_{x,i} + a_x \Delta t} \text{ ----- (1)}$$



The shaded area : $\Delta x \rightarrow$ displacement

$$x_f = x_i + \Delta x$$

Shaded Area $= \boxed{\Delta x = v_{x,i}\,\Delta t + \frac{1}{2}\Delta t\,\Delta v_x}$

$$\Delta x = v_{x,i}\,\Delta t + \frac{1}{2}\Delta t\,(a_x \Delta t)$$

$$\Delta x = v_{x,i}\,\Delta t + \frac{1}{2} a_x (\Delta t)^2$$

$$\boxed{x_f = x_i + v_{x,i}\,\Delta t + \frac{1}{2} a_x (\Delta t)^2} \text{ ---- (2)}$$

From $a_x = (v_{x,f} - v_{x,i})/\Delta t$, $\Delta t = \dfrac{v_{x,f} - v_{x,i}}{a_x}$ substitute above

$$x_f = x_i + v_{x,i}\,\frac{\cancel{v_{x,f}} - v_{x,i}}{a_x} + \frac{1}{2} a_x \frac{1}{a_x^2}\left[v_{x,f}^2 - 2\cancel{v_{x,i} v_{x,f}} + v_{x,i}^2\right]$$

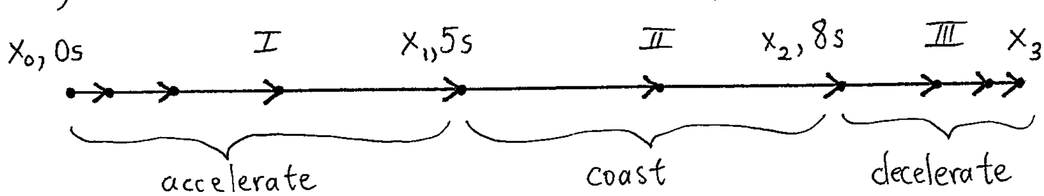
$$x_f = x_i + \frac{-v_{x,i}^2}{a_x} + \frac{1}{2a_x}\left(v_{x,f}^2 + v_{x,i}^2\right)$$

$$x_f = x_i + \frac{v_{x,f}^2 - v_{x,i}^2}{2a_x} \rightarrow 2a_x \Delta x = v_{x,f}^2 - v_{x,i}^2$$

$$\boxed{v_{x,f}^2 = v_{x,i}^2 + 2a_x \Delta x} \text{ ----(3)}$$

PROB. A rocket accelerates at 50 m/s² for 5 s, coasts for 3 s, then deploys a braking parachute and decelerates at 3 m/s² until complete stop.

a) What is the total distance traveled?



In Region I: (Motion w/ constant acceleration)

$$\boxed{x_1 = x_0 + v_{x,0}(t_1 - t_0) + \frac{1}{2} a_{x,1}(t_1 - t_0)^2}$$

$$x_1 = \frac{1}{2} 50 \frac{m}{s^2} (5s - 0s)^2$$

$$x_1 = 625 \ m$$

In Region II: (Uniform motion)

$$\boxed{x_2 = x_1 + v_{x,1} \Delta t}, \quad v_{x,1} = v_{x,0} + a_{x,1}\Delta t = 50 \frac{m}{s^2} 5s = 250 \frac{m}{s}$$

$$x_2 = 625 \ m + 250 \frac{m}{s} 3s$$

$$x_2 = 1375 \ m$$

In Region III: (Motion w/ constant deceleration) We don't know how long it takes for the sled to stop.

$$\boxed{v_{x,3}^2 = v_{x,2}^2 + 2a_{x,2}\Delta x}$$

$$= v_{x,2}^2 + 2a_{x,2}(x_3 - x_2)$$

Solve for x_3

$$x_3 = x_2 + \frac{v_{x,3}^2 - v_{x,2}^2}{2a_{x,2}}$$

$$x_3 = 1375 m + \frac{-(250 \frac{m}{s})^2}{2(-3)\frac{m}{s^2}}$$

$$x_3 = 11,800 \ m$$

2.5 Free Fall

The motion of an object moving under only the action of gravity is called free fall.

No air resistance. (Heavy objects, short distances)

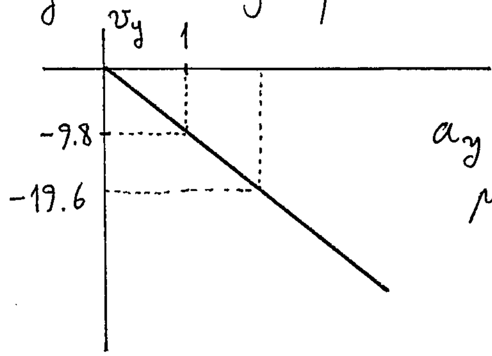
Galileo's Experiments: Neglecting air resistance, different falling objects hit the ground at the same time.

Any two objects in free fall, regardless of their mass, have the same acceleration $\vec{a}_{free\,fall}$.

$$\vec{a}_{freefal} = 9.80 \frac{m}{s^2} \text{ (vertically downward)} \rightarrow \text{on earth}$$

$$|\vec{a}_{free\,fall}| \equiv g$$

* g is always positive



$a_y = -g = -9.80\ \frac{m}{s^2}$

Motion w/ constant acceleration!

PROB. A rock is released from 200 m tall building.
a) How long does it take for it to impact ground?
b) What is its impact velocity?

y

$y_0 = 200\ m$
$v_0 = 0\ m/s$

$a_y = -g$

$y_1 = 0\ m$
$v_1 = ?$

a) $\boxed{y_1 = y_0 + v_0 \Delta t + \frac{1}{2} a_y \Delta t^2}$

$0\ m = 200\ m + \frac{1}{2}(-9.80\ \frac{m}{s^2})(\Delta t)^2$

$(+9.80\ \frac{m}{s^2})\ \Delta t^2 = 400\ m$

$\Delta t = \sqrt{\dfrac{400 m}{9.80\ m/s^2}}$

$\Delta t = \pm 4.52\ s$ (discard (-) Δt)

b) $\boxed{v_1 = v_0 + a_y \Delta t}$

$v_1 = -9.80\ \frac{m}{s^2} \times 4.52\ s = -44.3\ \frac{m}{s}$

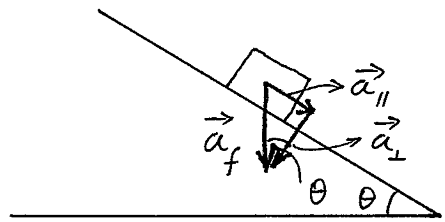
PROB. A cannonball is shot straight up at 100 m/s. How high does it go?

$\boxed{v_1^2 = v_2^2 - 2g\,\Delta y}$

$0 = (100\ \frac{m}{s})^2 - 2(9.80\ \frac{m}{s^2})\,\Delta y$

$\Delta y = 510\ m$

2.6 Motion on an Inclined plane



Ignore friction.

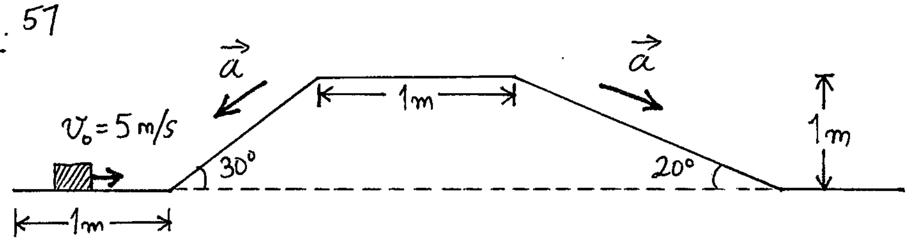
$\vec{a}_f = \vec{a}_{\parallel} + \vec{a}_{\perp}$

Surface of the incline "blocks" $\vec{a}_{\perp}$.

$\boxed{a_s = \pm g \sin\theta}$

If the surface is perfectly horizontal $\theta = 0°$, $a_s = 0$, the object is at rest if starts from rest.

PROB. 57



a) What is the speed of the puck as it goes over the top?

$\boxed{v_T^2 = v_0^2 - 2g\sin\theta\,\Delta l_1}$, $\Delta l_1 = 1m/\sin 30° = 2m$

$v_T^2 = (5\,\frac{m}{s})^2 - 2\,(9.80\,\frac{m}{s^2})\,\frac{1}{2}\,2m$

$v_T^2 = 25(\frac{m}{s})^2 - 19.6\,(\frac{m}{s})^2$

$v_T^2 = 5.4\,\frac{m^2}{s^2} \rightarrow v_T = +2.3\,\frac{m}{s}$ (Moving to the right)

b) What is its speed when it reaches the level track on the right side?

$\boxed{v_B^2 = v_T^2 + 2g\sin 20°\,\Delta l_2}$, $\Delta l_2 = 1m/\sin 20° = 2.92\ m$

$v_B^2 = 5.4\,\frac{m^2}{s^2} + 2\,(9.80\,\frac{m}{s^2})\,0.34 \times 2.92\,m$

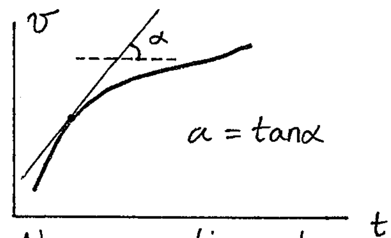
$v_B^2 = 5.4\,(\frac{m}{s})^2 + 19.6\,(\frac{m}{s})^2 = 25\,(\frac{m}{s})^2$

$v_B = +5\,\frac{m}{s}$ (Moving to the right)

2.7 Instantaneous Acceleration

Instantaneous acceleration is the slope of the line that is tangent to the v-t curve at time t.

$$a \equiv \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$



<u>Finding v if we know a:</u>

1) Divide the acceleration curve into N narrow time steps.
2) In each time step k, the change in velocity is $a_k \Delta t$.
3) This is the area under the curve during the step.
4) Add these up to get Δv. in the limit $\Delta t \to 0$.

$$v_f = v_i + \lim_{\Delta t \to 0} \sum_{k=1}^{N} a_k \Delta t$$

$$\boxed{v_f = v_i + \int_{t_i}^{t_f} a \, dt}$$

$v_f = v_i$ + Area under the acceleration curve between t_i and t_f.

<u>PROB.</u> 80 A realistic model of Olympic sprinters velocity is

$$v_x = a(1 - e^{-bt})$$

For 1987 World Champion, $a = 11.81 \frac{m}{s}$ and $b = 0.6887 \frac{1}{s}$.

a) What was this person's acceleration at $t = 0s, 2s, 4s$?

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(a - a e^{-bt})$$

$$\boxed{a_x = ab\, e^{-bt}}$$ (it starts high and then drops) as opposed to v_x, which started 0 and increased)

$$a_x(t=0s) = ab = 8.1335 \; m/s^2$$

$$a_x(t=2s) = 8.1335 \frac{m}{s^2} \; 0.2522 = 2.0515 \; m/s^2$$

$$a_x(t=4s) = 0.51747 \; m/s^2$$