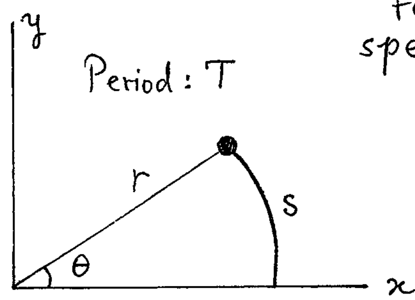
4.5 Uniform Circular Motion



For a particle moving at a constant speed around a circle of radius r,

$$\boxed{v = \frac{1 \text{ circumference}}{1 \text{ period}} = \frac{2\pi r}{T}}$$

* T is period, i.e. time it takes to complete circle.

* θ is angular position. It is [positive when measured counterclockwise (ccw) / negative when measured clockwise (cw)]

For example $-30°$ and $330°$ are the same angular position.

* s is arclength. $\theta(\text{rad}) = \frac{s}{r} \rightarrow \boxed{s = \theta(\text{rad})\, r}$ only when θ is measured in radians.

$$\theta_{\substack{\text{Full} \\ \text{Circle}}} = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ rad}$$

$$\boxed{1 \text{ rev} = 360° = 2\pi \text{ rad}} \rightarrow 1 \text{ rad} = 1 \text{rad}\, \frac{360°}{2\pi \text{rad}} = 57.3° \sim 60°$$

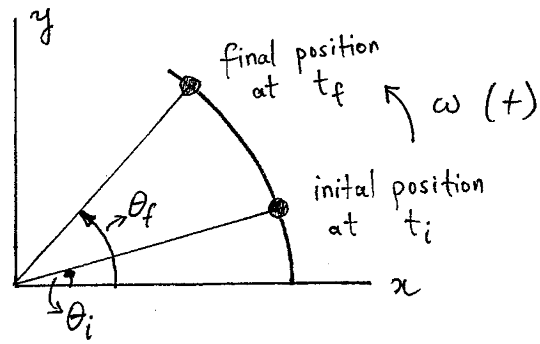
__Angular velocity (ω):__

Angular displacement $\Delta\theta$

$$\Delta\theta = \theta_f - \theta_i$$

Average angular velocity ω_{avg}

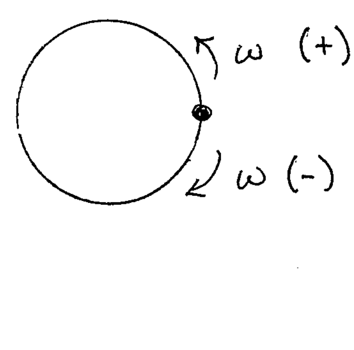
$$\omega_{avg} \equiv \frac{\Delta\theta}{\Delta t}$$



Instantenous velocity ω

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \rightarrow \text{Uniform circular motion when } \omega \text{ is constant.}$$

$$\begin{cases} \omega \text{ is positive for ccw rotation} \\ \omega \text{ is negative for cw rotation} \end{cases}$$



Drawing analogy to 1d motion,

$\omega \equiv$ slope of the θ vs. t graph
$\theta_f = \theta_i$ + area under ω vs. t graph b/w t_i and t_f

$$\boxed{\theta_f = \theta_i + \int_{t_i}^{t_f} \omega \, dt} \rightarrow \boxed{\theta_f = \theta_i + \omega \Delta t} \text{ when } \omega \text{ is constant}$$

The angular displacement is $\Delta\theta = 2\pi$ for a full period (T),

$$\boxed{\omega = \frac{2\pi \text{ rad}}{T}}$$

4.6 Velocity (v) and Acceleration (a) in Uniform Circular Motion

* Velocity vector $\vec{v}$ is always tangent to the circle.

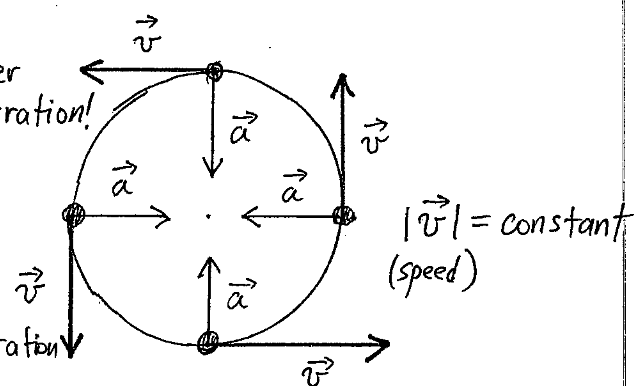
s: arc length $\rightarrow$ $v = \dfrac{ds}{dt} = \dfrac{d(r\theta)}{dt} = r\underbrace{\dfrac{d\theta}{dt}}_{\text{angular velocity}}$

$\boxed{v = r\omega}$ $[\omega] := \text{rad/s}$

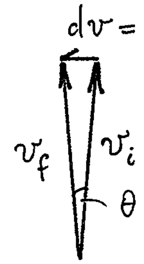
Also $v > 0$ for ccw motion
$v < 0$ for cw motion

* Acceleration $\vec{a}$ points to the center of the circle. Not constant acceleration!

$\vec{v}$ and $\vec{a}$ are always perpendicular at any point on the circle.



$\vec{a}$ is called centripetal acceleration



also $v = r\dfrac{d\theta}{dt} \rightarrow d\theta = \dfrac{v\,dt}{r}$

$dv = v\,d\theta \rightarrow d\theta = \dfrac{dv}{v}$

Equate

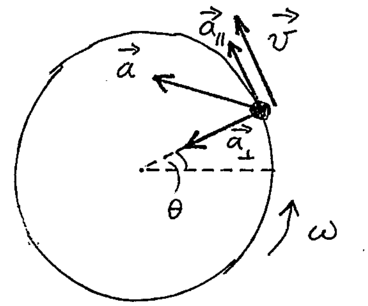
$$\frac{v\,dt}{r} = \frac{dv}{v} \rightarrow \boxed{\frac{dv}{dt} \equiv a = \frac{v^2}{r}} \text{ Centripetal Acceleration}$$

Also, $v = \omega r \rightarrow \boxed{a = \omega^2 r}$

4.7 Non-Uniform Circular Motion and Angular Acceleration

* Circular motion with changing speed.

Centripetal Acc. = Radial Acc. = $a_\perp$



1) Tangential component $\vec{a}_{\parallel}$ controls the speed of the particle

2) $\vec{a}_\perp$ is responsible for change of direction.

$$\vec{a} = \vec{a}_\perp + \vec{a}_{\parallel}$$

$$a = \sqrt{a_\perp^2 + a_{\parallel}^2} \rightarrow \text{magnitude of acceleration.}$$

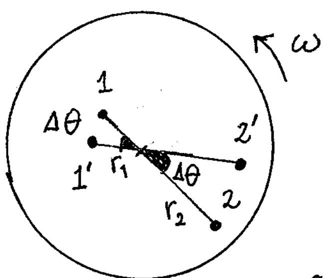
$a_{\parallel} = \dfrac{dv}{dt} \rightarrow$ $a_{\parallel}$ and v are both parellel to each other and tangent to the circle.

If $a_{\parallel}$ is constant $\rightarrow$

$$\boxed{\begin{aligned} s_f &= s_i + v_i \Delta t + \tfrac{1}{2} a_{\parallel} (\Delta t)^2 \\ v_f &= v_i + a_{\parallel} \Delta t \end{aligned}}$$

Angular Acceleration : α

Consider a rotating disc



* Points 1 and 2 have the same ω since they rotate by same $\Delta\theta$ in time Δt.

* They have different v since $2\pi r_1 \neq 2\pi r_2$

$$a_{\parallel} = \frac{dv}{dt} = \frac{d(r\omega)}{dt}$$

$$\boxed{a_{\parallel} = r\frac{d\omega}{dt} = r\alpha} \quad \text{where} \quad \boxed{\alpha \equiv \frac{d\omega}{dt}}$$

the rate at which angular velocity ω changes.

Units of α : $[\alpha] = \dfrac{\text{rad}}{s^2}$

→ Remember that sign of acceleration with respect to v is what determines whether the point is speeding up or slowing down.

Sign of α : $\left[\begin{array}{l}\text{positive if } \omega \text{ is increasing ccw } \underline{\text{OR}} \text{ decreasing cw} \\ \text{negative if } \omega \text{ is decreasing ccw } \underline{\text{OR}} \text{ increasing cw}\end{array}\right.$

* Two points on a rotating object have the <u>same</u> α, but <u>different</u> $a_{\parallel}$.

Because α is time derivative of ω:

1) α is the slope of ω vs. t graph.
2) $\omega_f = \omega_i$ + area under α vs. t graph b/w t_i and t_f.

Circular Kinematic Eqns. in terms of angular quantities:

$$\begin{cases} s_f = s_i + v_i \Delta t + \frac{1}{2} a_{\parallel} (\Delta t)^2 \\ v_f = v_i + a_{\parallel} \Delta t \end{cases}$$

Divide both sides w/ r and note that $\frac{s}{r} = \theta$, $\frac{v}{r} = \omega$ and $\frac{a_{\parallel}}{r} = \alpha$

$$\boxed{\begin{array}{l} \theta_f = \theta_i + \omega \Delta t + \frac{1}{2} \alpha (\Delta t)^2 \\ \omega_f = \omega_i + \alpha \Delta t \end{array}}$$

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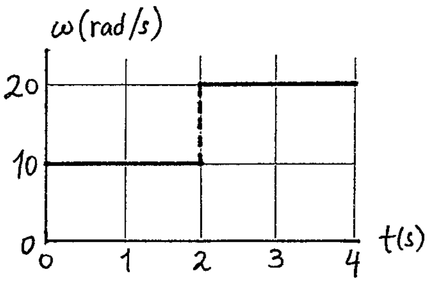

How many revolutions does the object make during the first 4 s?

$\Delta\theta$ = area under ω vs t graph.

$\Delta\theta = 10 \cdot 2 + 20 \cdot 2$

$\Delta\theta = 60$ rad

$1 \text{ rev} = 2\pi \text{ rad} \rightarrow \Delta\theta = 60 \text{ rad} = 60 \text{ rad} \frac{1 \text{ rev}}{2\pi \text{ rad}} = \underline{9.55 \text{ rev}}$