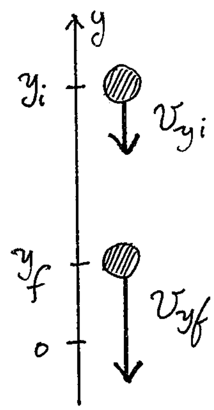
CHAPTER 10

* Energy as "natural money".

10.2 Kinetic Energy and Gravitational Potential Energy

Consider an object in free fall.



$$v_{yf}^2 = v_{yi}^2 - 2g(y_f - y_i)$$

$$\underbrace{v_{yf}^2 + 2gy_f}_{\text{Before}} = \underbrace{v_{yi}^2 + 2gy_i}_{\text{After}}$$

* For free fall $\boxed{v^2 + 2gy}$ is a conserved quantity.

* To generalize for any kind of motion, use Newton's Laws:

$$F_{net,y} = ma_y = m\frac{dv_y}{dt} \quad \text{and} \quad F_{net,y} = -mg$$

$$-mg = m\frac{dv_y}{dt} = m\frac{dv_y}{dy}\underbrace{\frac{dy}{dt}}_{= v_y} \rightarrow \text{Chain Rule for differentiation.}$$

$$-mg = mv_y\frac{dv_y}{dy} \quad \text{Multiply both sides by } dy$$

$$-mg\,dy = mv_y\,dv_y \quad \text{Now integrate both sides}$$

$$-mg\int_{y_i}^{y_f} dy = m\int_{v_i}^{v_f} v_y\,dv_y$$

$$-mgy_f + mgy_i = m\frac{v_{yf}^2}{2} - m\frac{v_{yi}^2}{2}$$

$$\boxed{\frac{1}{2}mv_{yi}^2 + mgy_i = \frac{1}{2}mv_{yf}^2 + mgy_f}$$

Definition : Kinetic Energy : $K = \frac{1}{2} m v^2$

Definition : Gravitational Potential Energy: $U_g = mgy$

* Energy is a <u>scalar</u> quantity!
* Kinetic Energy depends on speed $|\vec{v}|$, not $\vec{v}$.
* K can never be negative. $K \geq 0$.

Unit of K → $[K] = kg\left(\frac{m}{s}\right)^2$ = Joule

Conservation of Energy : $\boxed{K_f + U_{gf} = K_i + U_{gi}}$

<u>Important</u>: The value of U_g depends on the choice of origin. It is not U_g but $\Delta U_g = U_{gf} - U_{gi}$ that is physically meaningful.

Just recall how we got $U_g = mgy$

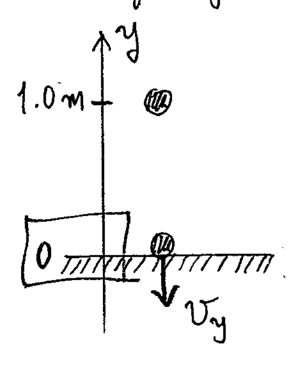
$$-mg \int_{y_i}^{y_f} dy = -mg(y_f - y_i) \equiv \Delta U_g$$

* We called $-mgy \equiv U_g$
* We might have called $-mgy + C \equiv U_g$ and retain definition of ΔU_g.

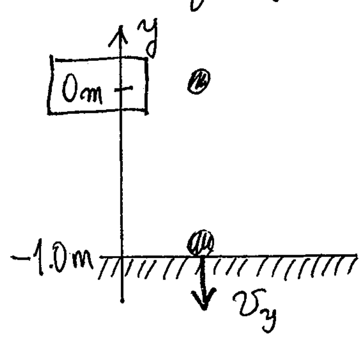
<u>EXAMPLE</u>

A rock is dropped from 1.0 m above the ground. If $m = 1.0$ kg, what is its speed when it hits the ground?

<u>Choice of origin #1</u>



<u>Choice of origin #2</u>



#1

$$K_i + U_{gi} = K_f + U_{gf}$$

$$1.0\,\text{kg}\left(9.80\,\tfrac{\text{m}}{\text{s}^2}\right)\,1.0\,\text{m} = \tfrac{1}{2}(1.0\,\text{kg})\,v_y^2$$

$$\boxed{v_y = 4.4\ \text{m/s}}$$

#2

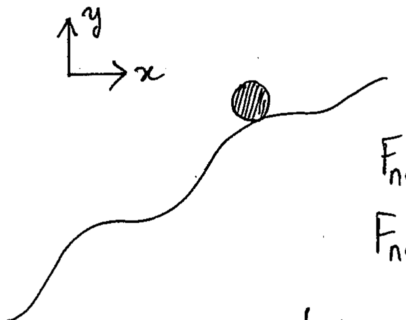
$$K_i + U_{gi} = K_f + U_{gf}$$

$$0 = \tfrac{1}{2}(1.0\,\text{kg})\,v_f^2 + 1.0\,\text{kg}\left(9.80\,\tfrac{\text{m}}{\text{s}^2}\right)(-1.0\,\text{m})$$

$$1.0\,\text{kg}\left(9.80\,\tfrac{\text{m}}{\text{s}^2}\right)\,1.0\,\text{m} = \tfrac{1}{2}(1.0\,\text{kg})\,v_f^2$$

$$\boxed{v_y = 4.4\ \text{m/s}}$$

10.3 A Closer Look at Gravitational Potential Energy



Write Newton's 2nd Law for x and y directions

$$F_{net,x} = F_x = \frac{dp_x}{dt} = m\frac{dv_x}{dt} = m\frac{dv_x}{dx}\frac{dx}{dt}$$

$$F_{net,y} = F_y = \frac{dp_y}{dt} = m\frac{dv_y}{dt} = m\frac{dv_y}{dy}\frac{dy}{dt}$$

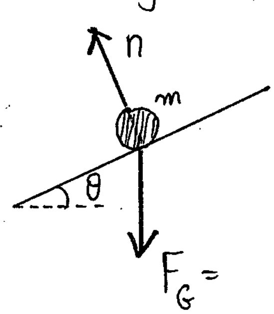
$$F_x = m v_x \frac{dv_x}{dx} \rightarrow \int_{x_i}^{x_f} F_x\,dx = \int_{v_{xi}}^{v_{xf}} m v_x\,dv_x$$

$$F_y = m v_y \frac{dv_y}{dy} \rightarrow \int_{y_i}^{y_f} F_y\,dy = \int_{v_{yi}}^{v_{yf}} m v_y\,dv_y$$

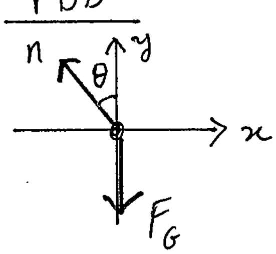
Which gives, upon adding two equations side by side,

$$\int_{x_i}^{x_f} F_x\,dx + \int_{y_i}^{y_f} F_y\,dy = \frac{1}{2}m\left(v_{xf}^2 - v_{xi}^2\right) + \frac{1}{2}m\left(v_{yf}^2 - v_{yi}^2\right)$$

To evaluate left hand side, draw the FBD for object on an infinitesimally small segment on the surface



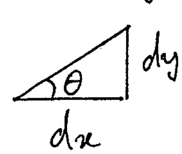
FBD



$$F_x = -n\sin\theta$$

$$F_y = n\cos\theta - mg$$

Also

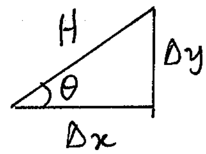


The equation then becomes,

$$-n\sin\theta\int_{x_i}^{x_f} dx + (n\cos\theta - mg)\int_{y_i}^{y_f} dy = \frac{1}{2}m\underbrace{\left(v_{xf}^2 + v_{yf}^2\right)}_{=v_f^2} + \frac{m}{2}\underbrace{\left(v_{xi}^2 + v_{yi}^2\right)}_{=v_i^2}$$

$$-n\sin\theta\,\Delta x + n\cos\theta\,\Delta y - mg\,\Delta y = \frac{1}{2}mv_i^2 - \frac{1}{2}mv_f^2$$

$$n\underbrace{\left(\cos\theta\,\Delta y - \sin\theta\,\Delta x\right)}_{\equiv 0} - mg\,\Delta y = \frac{1}{2}mv_i^2 - \frac{1}{2}mv_f^2$$



$$\left.\begin{aligned}\cos\theta\,\Delta y &= \frac{\Delta x}{H}\Delta y \\ \sin\theta\,\Delta x &= \frac{\Delta y}{H}\Delta x\end{aligned}\right\}\text{ identical!}$$

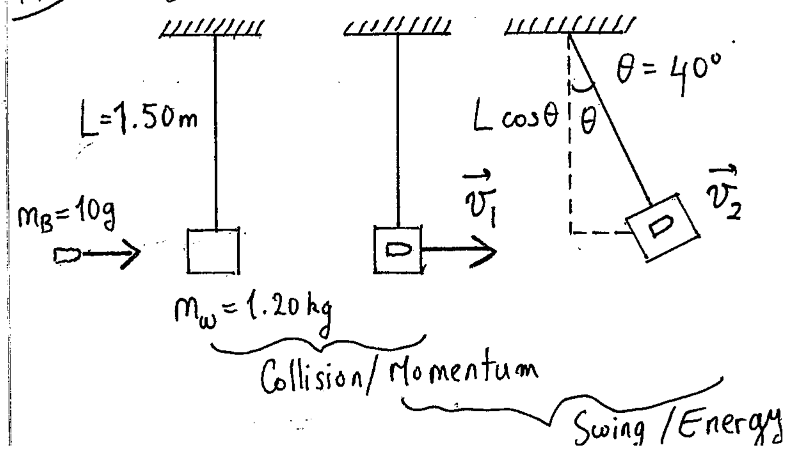
$$\boxed{\frac{1}{2}mv_i^2 + mgy_i = \frac{1}{2}mv_f^2 + mgy_f}$$ Conservation of Energy

Definition : Mechanical Energy : $E_{mech} = K + U$

* Mechanical Energy is Conserved : $\boxed{\Delta E_{mech} = \Delta K + \Delta U = 0}$
* Note that $\Delta K \neq \Delta U$.
* Consider a box sliding on the floor to a halt. $\Delta E_{mech} \neq 0$.
  There are cases in which Mechanical Energy E_{mech} is not conserved.
* Kinetic Energy and Potential Energy can be transformed into one another.

PROB Ballistic Pendulum

If θ is $40°$, what was the initial speed of the bullet?



Collision/Momentum Stage

Momentum is Conserved : $P_i = P_f$

$$m_B (v_{xi})_B + m_w \cancel{(v_{xi})_w}^{\,0} = (m_B + m_w)(v_{xf})_{B+w}$$

$$\boxed{(v_{xi})_B = \frac{m_B + m_w}{m_B} (v_{xf})_{B+w}}$$ From conservation of momentum

Swing Stage

Mechanical Energy is Conserved: $K_i + U_{gi} = K_f + U_{gf}$

$$\frac{1}{2}(m_B + m_w) v_1^2 + (m_B + m_w) g \cancel{y_1}^{\,0}$$
$$= \frac{1}{2}(m_B + m_w) \cancel{v_2^2}^{\,0} + (m_B + m_w) g y_2$$

where $v_1 = (v_{xf})_{B+w}$.

$$\frac{1}{2} v_1^2 = g y_2 \rightarrow \boxed{v_1 = \sqrt{2 g y_2}} = (v_{xf})_{B+w}$$

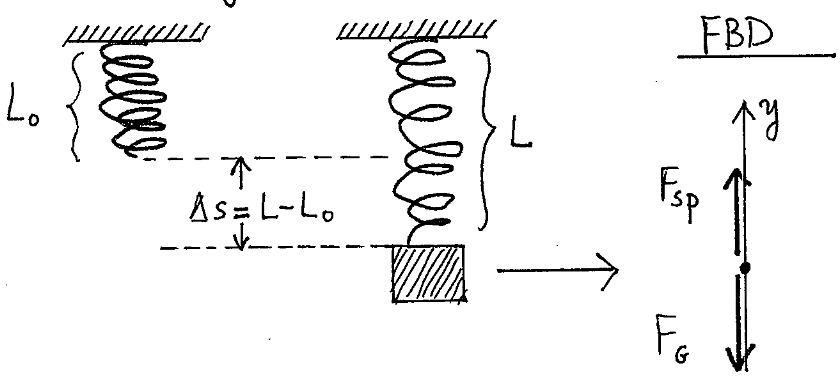
Also, $y_2 = L - L\cos\theta = L(1 - \cos\theta)$

Substituting in $(v_{xi})_B$ gives,

$$\boxed{(v_{xi})_B = \frac{m_B + m_w}{m_B} \sqrt{2 g L (1 - \cos\theta)}}$$

$$(v_{xi})_B = 320 \text{ m/s}$$

10.4 Restoring Forces and Hooke's Law



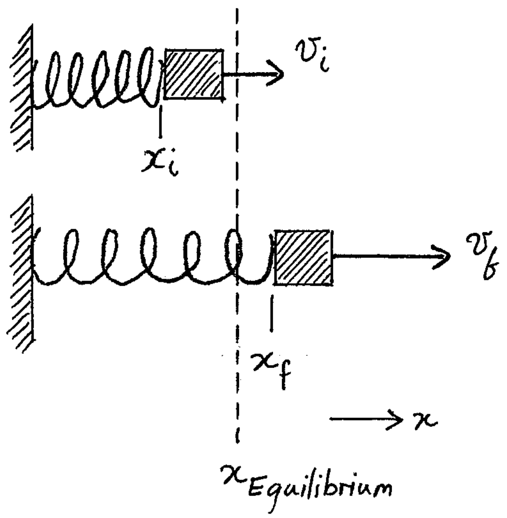
Restoring Force

$$(F_{sp})_s = -k \Delta s$$

(Hooke's Law)

* The sign of F_{sp} is always apposite to that of Δs.

10.5 Elastic Potential Energy



$$F_{net,x} = \frac{dp_x}{dt} = m\frac{dv_x}{dt} \quad \text{Newton's 2nd Law}$$

Hooke's Law: $F_{net,x} = -k\,\Delta x$

Δx is displacement from equilibrium position of the spring.

$$\Delta x = x - x_E$$

$$F_{net,x} = -k(x - x_E)$$

Newton's 2nd Law:

$$m\frac{dv_x}{dt} = -k(x - x_E)$$

$$m\frac{dv_x}{dx}\underbrace{\frac{dx}{dt}}_{=v_x} = -k(x - x_E)$$

$$m v_x \frac{dv_x}{dx} = -k(x - x_E)$$

$$m v_x\, dv_x = -k(x - x_E)\,dx$$

Integrate both sides as usual to get:

$$m\int_{v_{xi}}^{v_{xf}} v_x\, dv_x = -k\int_{x_i}^{x_f}(x - x_E)\,dx$$

$$\frac{1}{2}m\left(v_{xf}^2 - v_{xi}^2\right) = -k\left[\frac{1}{2}\left(x_f^2 - x_i^2\right) - x_E\left(x_f - x_i\right)\right]$$

$$= -\frac{k}{2}\left[x_f^2 - x_i^2 - 2x_E x_f + 2x_E x_i + x_E^2 - x_E^2\right]$$

* Alternatively, we might have made the change of variables $u = x - x_E$ to carry out the integration

$$\frac{1}{2}m\left(v_{xf}^2 - v_{xi}^2\right) = -\frac{k}{2}\Big[\underbrace{(x_f - x_E)}_{=\Delta x_f}{}^2 - \underbrace{(x_i - x_E)}_{=\Delta x_i}{}^2\Big]$$

$$\boxed{\frac{1}{2}m v_{xf}^2 + \frac{1}{2}k\,\Delta x_f^2 = \frac{1}{2}m v_{xi}^2 + \frac{1}{2}k\,\Delta x_i^2}$$

Note that the conserved quantity is $\boxed{\frac{1}{2}mv^2 + \frac{1}{2}k(\Delta x)^2}$

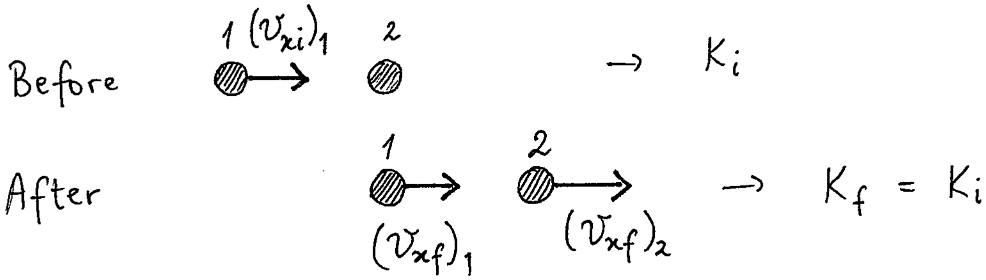
Define: Elastic Potential Energy: $U_s = \frac{1}{2}k(\Delta x)^2$

Conservation of Energy: $\boxed{K_i + U_{si} = K_f + U_{sf}}$

* For a system that has both elastic and gravitational potential energy: $E_{mech} = K + U_g + U_s$

10.6 Elastic Collisions

* A collision in which mechanical energy is conserved is called a perfectly <u>elastic collision</u>.



* Conservation of momentum → obeyed in <u>any collision</u>.
* Conservation of energy → because the collision is <u>elastic</u>.

1) Conservation of momentum : $\boxed{P_{xi} = P_{xf}}$

$$m_1 (v_{xi})_1 = m_1 (v_{xf})_1 + m_2 (v_{xf})_2$$

2) Conservation of energy : $\boxed{K_i = K_f}$

$$\frac{1}{2} m_1 (v_{xi})_1^2 = \frac{1}{2} m_1 (v_{xf})_1^2 + \frac{1}{2} m_2 (v_{xf})_2^2$$

Solving for $(v_{xf})_1$ from Eq (1)

$$(v_{xf})_1 = (v_{xi})_1 - \frac{m_2}{m_1} (v_{xf})_2$$

Substituting this in Eq. (2)

$$\cancel{\frac{1}{2} m_1 (v_{xi})_1^2} = \frac{1}{2} m_1 \left[(v_{xi})_1 - \frac{m_2}{m_1} (v_{xf})_2 \right]^2 + \frac{1}{2} m_2 (v_{xf})_2^2$$

$$= \frac{1}{2} m_1 \left[\cancel{(v_{xi})_1^2} - 2 \frac{m_2}{m_1} (v_{xi})_1 (v_{xf})_2 + \left(\frac{m_2}{m_1}\right)^2 (v_{xf})_2^2 \right]$$

$$+ \frac{1}{2} m_2 (v_{xf})_2^2$$

Divide both sides by (m_2/m_1),

$$0 = \frac{1}{2} m_1 \left[-2 (v_{xi})_1 (v_{xf})_2 + \frac{m_2}{m_1} (v_{xf})_2^2 \right] + \frac{1}{2} m_1 (v_{xf})_2^2$$

$$0 = \cancel{\frac{1}{2} m_1} \left[-2 (v_{xi})_1 (v_{xf})_2 + \left(1 + \frac{m_2}{m_1}\right) (v_{xf})_2^2 \right]$$

$$0 = (v_{xf})_2 \left[\left(1 + \frac{m_2}{m_1}\right) (v_{xf})_2 - 2 (v_{xi})_1 \right]$$

Solutions are $(v_{xf})_2 = 0$ (not interesting) and

$$\boxed{(v_{xf})_2 = \frac{2 (v_{xi})_1}{1 + \frac{m_2}{m_1}} = \frac{2 m_1}{m_1 + m_2} (v_{xi})_1}$$

This expression for $(v_{xf})_2$ can be substituted in Eq. (1)

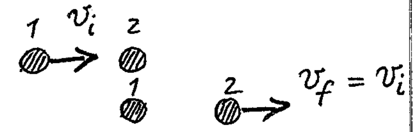
$$(v_{xf})_1 = (v_{xi})_1 - \frac{m_2}{\cancel{m_1}}\,\frac{2\cancel{m_1}}{m_1+m_2}\,(v_{xi})_1$$

$$\boxed{\begin{aligned}(v_{xf})_1 &= \frac{m_1-m_2}{m_1+m_2}\,(v_{xi})_1\\ (v_{xf})_2 &= \frac{2m_1}{m_1+m_2}\,(v_{xi})_1\end{aligned}}$$

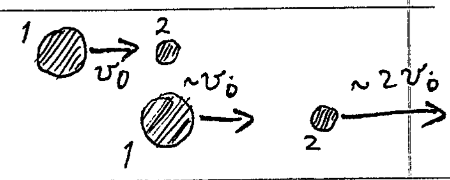
Collision in which ball #2 was initially at rest.

Consider three distinct cases:

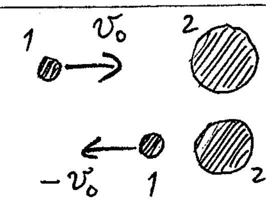
1) $m_1 = m_2 \rightarrow \begin{cases}(v_{xf})_1 = 0\\ (v_{xf})_2 = (v_{xi})_1\end{cases}$



2) $m_1 \gg m_2 \rightarrow \begin{cases}(v_{xf})_1 \approx (v_{xi})_1\\ (v_{xf})_2 \approx 2(v_{xi})_1\end{cases}$

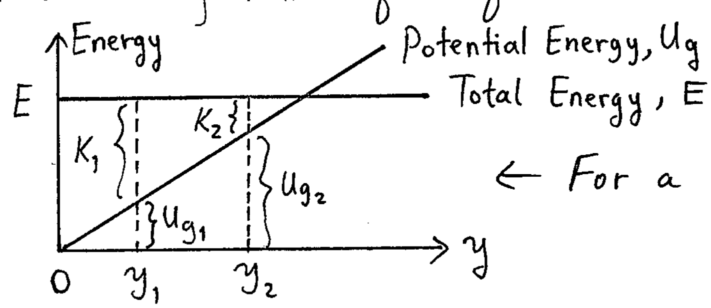


3) $m_1 \ll m_2 \rightarrow \begin{cases}(v_{xf})_1 = -(v_{xi})_1\\ (v_{xf})_2 \approx 0\end{cases}$



10.7 Energy Diagrams

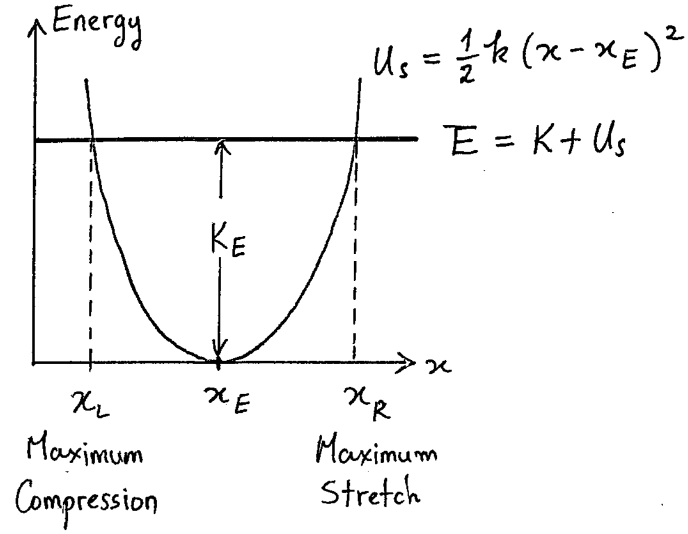
* For an object in a free fall



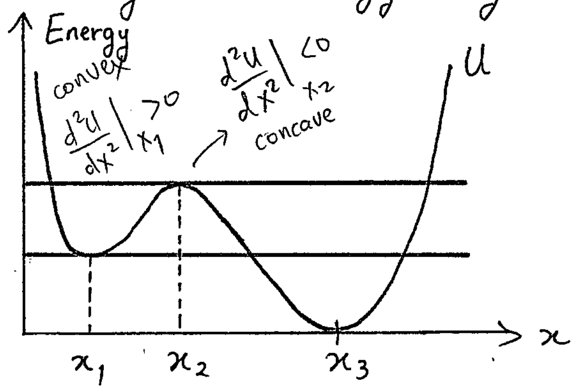
$$\boxed{E = K + U_g}$$

← For a ball tossed upward.

* For a mass on a horizontal spring:



* A more general Energy Diagram

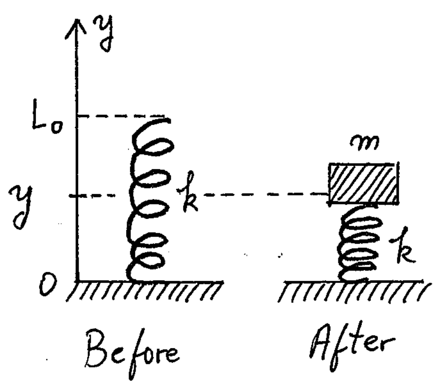


X_1 and $X_3 \rightarrow$ Stable Equilibrium

$X_2 \rightarrow$ Unstable Equilibrium

<u>The equilibrium point</u> is where the potential is <u>extremum</u>. (min <u>or</u> max)

<u>PROB</u>



What is the length of the compressed spring?

The total Potential Energy is

$$U_{TOT} = U_g + U_s$$

$$U_{TOT} = mgy + \frac{1}{2}k(y - L_0)^2$$

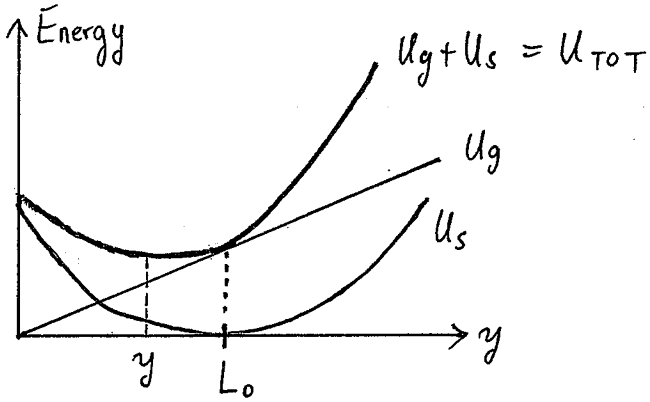
The equilibrium position is where U_{TOT} is <u>minimum</u>.

$\frac{dU_{TOT}}{dy} = 0$ gives y at which U_{TOT} is minimum.

$$\frac{dU_{TOT}}{dy} = \frac{d}{dy}\left(mgy + \frac{1}{2}k(y-L_0)^2\right) = 0$$

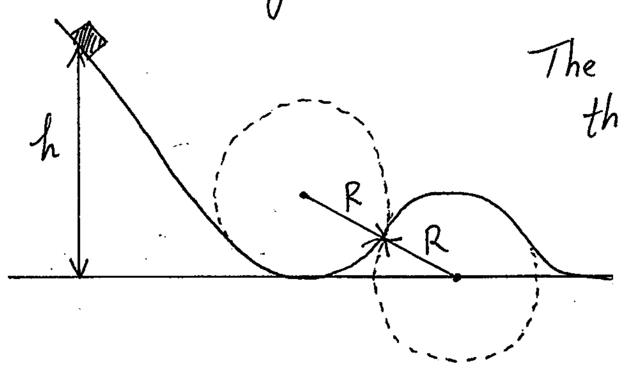
$$mg + \frac{1}{2}k\,2\,(y-L_0) = 0$$

$$\boxed{y = L_0 - \frac{mg}{k}}$$

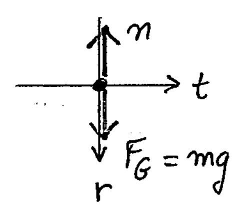


PROB 53

What is the maximum height h_{max} for the car to go over the hill w/o flying off?



The maximum speed for going over the hill safely is:



$$\sum F_r = mg - n = m\frac{v^2}{R}$$

The car stops executing circular motion and flies off when $\underline{n=0}$.

$$mg = m\frac{v^2}{R} \rightarrow \boxed{v_{max} = \sqrt{gR}}$$

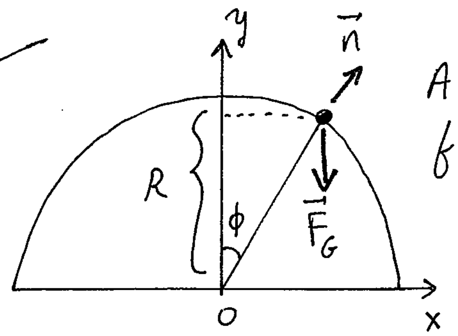
From conservation of energy:

$$\frac{1}{2}m\cancelto{0}{v_0^2} + mgh_{max} = \frac{1}{2}mv_{max}^2 + mgR$$

$$g h_{max} = \frac{v_{max}^2}{2} + g R$$

$$h_{max} = \frac{gR}{2g} + R = \boxed{\frac{3}{2} R}$$

PROB 76



A sled starts from the top of a frictionless, hemispherical snow-covered hill seen on the left.

a) $v(\phi) = ?$

b) Maximum speed the sled can have without sliding off.

c) At what angle does the sled fly-off?

a) From conservation of energy:

$$K_i + U_{g_i} = K_f + U_{g_f}$$

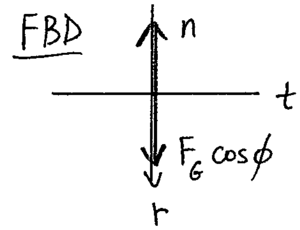
$$\frac{1}{2} m v_i^2 + m g y_i = \frac{1}{2} m v_f^2 + m g y_f$$

$$0 + \cancel{m} g R = \frac{1}{2} \cancel{m} v^2 + \cancel{m} g R \cos\phi$$

$$2 g R (1 - \cos\phi) = v^2$$

$$\boxed{v = \sqrt{2 g R (1 - \cos\phi)}}$$

b) Use Newton's 2nd Law



$$(F_{net})_r = F_G \cos\phi - n = m \frac{v^2}{R} = m a_r$$

$$m g \cos\phi - m \frac{v^2}{R} = n$$

The maximum speed before flying off is when $n = 0$,

$$\cancel{m} g \cos\phi = \cancel{m} \frac{v_{max}^2}{R}$$

$$\boxed{v_{max} = \sqrt{g R \cos\phi}}$$

c) The sled flies off at ϕ_{max} when $v = v_{max}$,

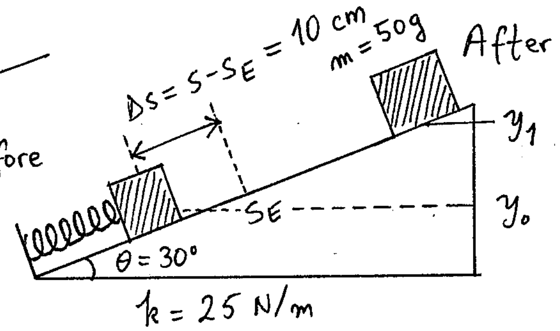
$$v_{max} = \sqrt{2 g R (1 - \cos\phi_{max})}$$

$$\sqrt{gR\cos\phi_{max}} = \sqrt{2gR(1-\cos\phi_{max})}$$

$$\cos\phi_{max} = 2 - 2\cos\phi_{max}$$

$$\cos\phi_{max} = \frac{2}{3} \rightarrow \boxed{\phi_{max} = 48^\circ}.$$

PROB 41



What distance along the incline does the box travel?

The total mechanical energy is conserved.

$$E_{mech} = K + U_{g_i} + U_s$$

$$K_i + (U_g)_i + (U_s)_i = K_f + (U_g)_f + (U_s)_f$$

$$\frac{1}{2}mv_0^2 + mgy_0 + \frac{1}{2}k(\Delta s)_0^2 = \frac{1}{2}mv_1^2 + mgy_2 + \frac{1}{2}k\cancel{(\Delta s)_1^2}^{\,0}$$

Note that, $v_0 = 0 \frac{m}{s}$, $y_0 = 0.0$ m, $v_1 = 0.0 \frac{m}{s}$, hence

$$\frac{1}{2}k(\Delta s)^2 = mgy_2$$

$$\frac{1}{2}\, 25\frac{N}{m}\,(0.1m)^2 = (0.05\,kg)(9.80\,\frac{m}{s^2})\,y_2$$

$$y_2 = 0.25\,m$$

Note that we didn't need to know K.