CHAPTER 12

* Particle model vs. Rigid Body model
* Basic types of motion for a rigid body
    1) Translational Motion
    2) Rotational Motion
    3) Combination of 1 + 2.

Rotational Kinematics.

Angular velocity : $\omega = \dfrac{d\theta}{dt}$ ($\omega > 0$ for ccw motion)

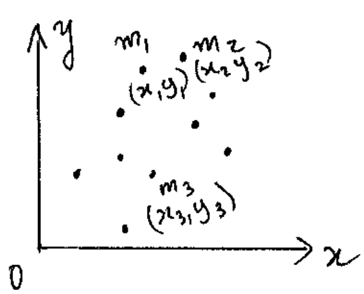
Angular acceleration : $\alpha = \dfrac{d\omega}{dt}$

$\omega > 0$	$\omega > 0$	$\omega < 0$	$\omega < 0$
$\alpha > 0$	$\alpha < 0$	$\alpha > 0$	$\alpha < 0$
speeding up ccw	slowing down ccw	slowing down cw	speeding up cw

$$\boxed{\begin{array}{ll} v_r = 0 & a_r = \dfrac{v^2}{r} = \omega^2 r \\ v_t = r\omega & a_t = r\alpha \end{array}}$$

12.2 Rotation about the center of mass

An unconstrained object on which there is no net Force, rotates about a point called "center of mass".



$$\begin{cases} x_{cm} = \dfrac{m_1 x_1 + m_2 x_2 + \cdots}{m_1 + m_2 + \cdots} = \dfrac{1}{M} \sum\limits_i m_i x_i \\ y_{cm} = \dfrac{m_1 y_1 + m_2 y_2 + \cdots}{m_1 + m_2 + \cdots} = \dfrac{1}{M} \sum\limits_i m_i y_i \end{cases}$$

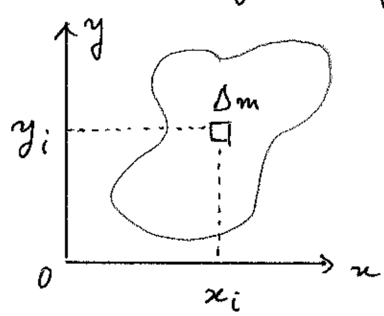
(x_{cm}, y_{cm}) are the center of mass coordinates.

* When $m_1 = m_2 = m_3 = \cdots = m$ for N particles,

$$x_{cm} = \frac{m(x_1 + x_2 + \cdots)}{Nm} = \frac{x_1 + x_2 + \cdots}{N} \rightarrow \text{Average } x\text{-coordinate}$$

$$y_{cm} = \frac{m(y_1 + y_2 + \cdots)}{Nm} = \frac{y_1 + y_2 + \cdots}{N} \rightarrow \text{Average } y\text{-coordinate}$$

* For a Rigid Body of mass M,



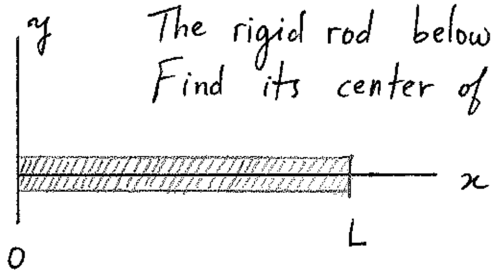
$$x_{cm} = \frac{1}{M}\sum_{i}^{N} x_i \,\Delta m \quad \underline{\text{and}} \quad y_{cm} = \frac{1}{M}\sum_{i}^{N} y_i \,\Delta m$$

As $\Delta m \to 0$ and $N \to \infty$

$$\boxed{x_{cm} = \frac{1}{m}\int x\, dm \quad \underline{\text{and}} \quad y_{cm} = \frac{1}{M}\int y\, dm}$$

<u>Example</u>

The rigid rod below is of length L and mass M. Find its center of mass.



The density per length: $\lambda = \frac{M}{L}$

$$dm = \lambda\, dx = \frac{M}{L}\, dx$$

$$x_{cm} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^L x \frac{M}{L}\, dx$$

$$x_{cm} = \frac{1}{L}\int_0^L x\, dx = \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L$$

$\boxed{x_{cm} = \frac{L}{2}}$ as expected.

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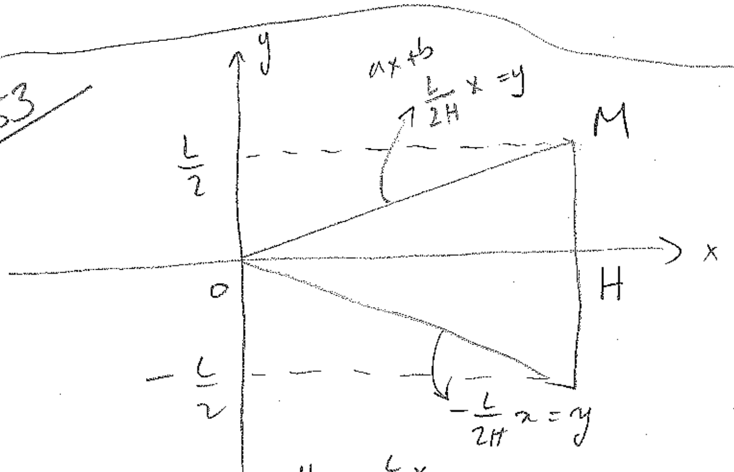


$$x_{cm} = \frac{1}{M}\int x\, dm$$

$$dm = \sigma\, dA = \frac{M}{A}\, dx\, dy$$

$$dm = \frac{2M}{HL}\, dx\, dy$$

For $L = 20$ cm and $H = 30$ cm

$$\begin{cases} x_{cm} = \frac{2}{3}\, 30\text{ cm} = 20\text{ cm} \\ y_{cm} = 0\text{ cm} \end{cases}$$

$$y_{cm} = \frac{1}{M}\int_0^H \left(\int_{-\frac{L}{2H}x}^{+\frac{L}{2H}x} y\, \frac{2M}{HL}\, dy \right) dx$$

$$y_{cm} = \frac{2M}{MHL2}\int_0^H 0\, dy = 0$$

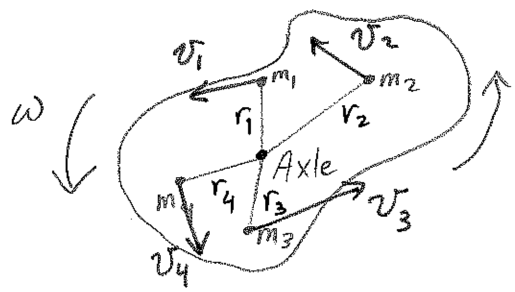
$$x_{cm} = \int_{-\frac{L}{2H}x}^{\frac{L}{2H}x} \underbrace{\int_0^H x\, \frac{2M}{HL}\, dx}\, dy = \int_0^H \left(\int_{-\frac{L}{2H}x}^{\frac{L}{2H}x} dy \right) \frac{2M}{HL}\, x\, dx$$

$$x_{cm} = \frac{1}{M}\frac{2M}{H\cancel{L}}\int_0^H \frac{\cancel{L}}{H}\, x\, x\, dx = \frac{1}{M}\frac{2M}{H^2 3}\, x^3 \Big|_0^H = \frac{2H}{3} = \frac{2}{3}H$$

12.3 Rotational Energy

The kinetic energy due to rotation is called rotational kinetic energy.

For a Rigid object:



$$K_{rot} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \cdots$$

$$K_{rot} = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \cdots$$

$$= \frac{1}{2} \underbrace{\left(\sum_i m_i r_i^2 \right)}_{\equiv I \,\to\, \text{Moment of Inertia}} \omega^2$$

Remember that $v = r\omega$

$$\boxed{K_{rot} = \frac{1}{2} I \omega^2}$$

* Note that $\frac{1}{2} I \omega^2$ is analogous to $\frac{1}{2} m v^2$.
* I depends on the <u>axis of rotation</u>.
* I depends on how the mass of the object is <u>distributed</u>.
* If the rotation axis (axle) is not through the center of mass of the object

$$\boxed{E_{mech} = K_{rot} + U_g = \frac{1}{2} I \omega^2 + M g y_{cm}}$$

* For a continuous distribution of mass, i.e. a rigid body

$$\boxed{I = \int r^2 \, dm}$$

<u>Example</u> Calculate the moment of inertia of a uniform rigid rod of mass M and length L that <u>rotates about a pivot at one end.</u>

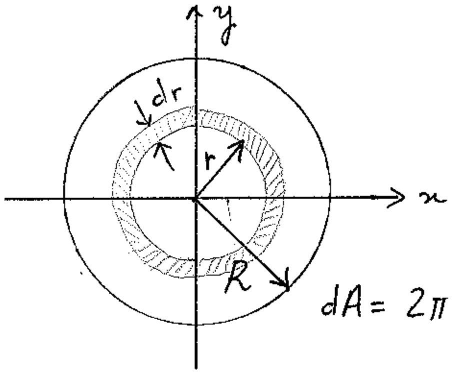
Density $\lambda = \frac{M}{L} \to dm = \frac{M}{L} dx$

$$I = \int x^2\, dm = \int_0^L x^2 \frac{M}{L}\, dx = \frac{M}{L}\int_0^L x^2\, dx$$

$$I = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \boxed{\frac{1}{3} M L^2}$$

Example

Find the moment of inertia for a circular disc of radius R and mass m that rotates on an axis passing through its center.



$dA = 2\pi r\, dr$

Density per unit area $\sigma = \dfrac{M}{\pi R^2}$

$$dm = \sigma\, dA = \frac{M}{\pi R^2}\, 2\pi r\, dr$$

$$I_{disc} = \int r^2\, dm$$

$$I_{disc} = \int_0^R r^2 \frac{M}{\pi R^2}\, 2\pi r\, dr$$

$$I_{disc} = \frac{2M}{R^2}\int_0^R r^3\, dr$$

$$= \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \boxed{\frac{1}{2} M R^2}$$

The parallel axis theorem

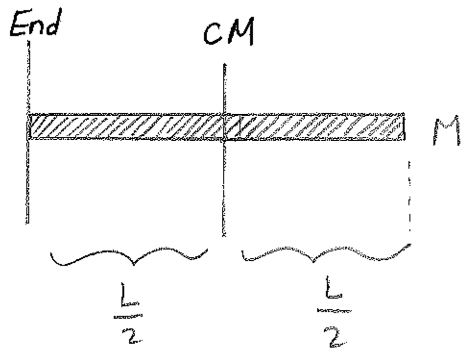
If the axis of interest is a distance d from a parallel axis through the center of mass, the moment of inertia

$$\boxed{I = I_{cm} + M d^2}$$

Example

Calculate the moment of inertia of a uniform rigid rod of mass M and Length L that rotates about a pivot through its center of mass?

We have already calculated I for a rod rotating about a pivot at one end.



$$I_{end} = \frac{1}{3} M L^2$$

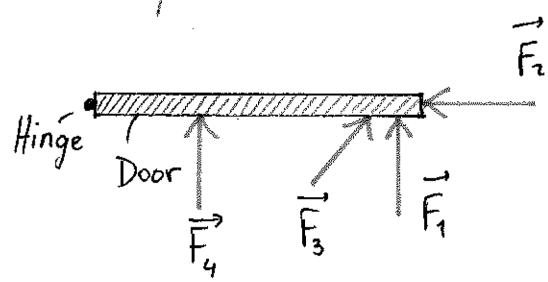
$$I_{end} = I_{CM} + M d^2$$

$$I_{end} = I_{CM} + M \left(\frac{L}{2}\right)^2$$

$$\frac{1}{3} M L^2 = I_{CM} + \frac{1}{4} M L^2$$

$$I_{CM} = \left(\frac{1}{3} - \frac{1}{4}\right) M L^2 = \boxed{\frac{1}{12} M L^2}$$

12.5 Torque



All $\vec{F}_1$, $\vec{F}_3$ and $\vec{F}_4$ will open the door. $\vec{F}_2$ will not.

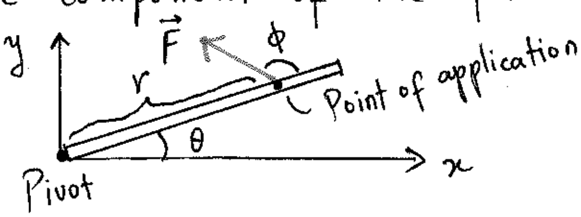
$\vec{F}_1$ will open it the most easily.

$\vec{F}_4$ opens it but not as easy as $\vec{F}_1$.

The ability of a force to cause rotation depens on:

1) Its magnitude.
2) The distance r from the point of application to the pivot point.
3) The angle.

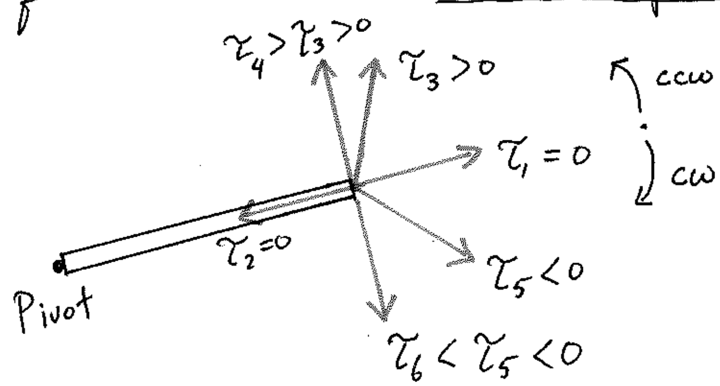
Combining 1 & 3 we can say that it depens on the magnitude of the component of the force in the direction of rotation.



Torque is defined as

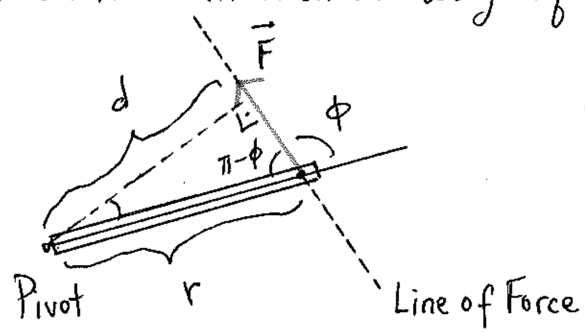
$$\boxed{\tau = r F \sin\phi}$$

* τ is the rotational equivalent of Force.
* Units of τ is $N \cdot m$ in SI units.
* τ is a vector like F. We'll use its sign to denote direction:
    1) $\tau > 0$ for torque causing ccw rotation
    2) $\tau < 0$ for torque causing cw rotation
* Torque is calculated <u>about a pivot point.</u>



Alternatively, $\boxed{\tau = r F_t}$ where F_t is the component of force perpendicular to the radial line. $\boxed{F_t = F \sin\phi.}$

Yet another alternative way of looking at τ;



$d = r\sin(\pi - \phi) = r\sin\phi$

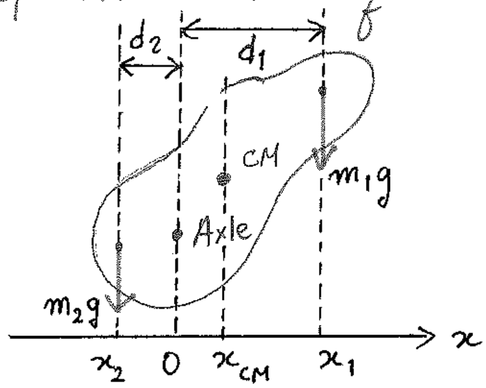
$$\boxed{|\tau| = d\,F = F r \sin\phi}$$

d is the distance from the pivot point to the line of $\vec{F}$.

* d is called "moment arm".

Net torque : $\tau_{net} = \tau_1 + \tau_2 + \dots = \sum_i \tau_i$

Gravitational Torque : Gravitational Force acting on every particle making up the object is $\boxed{F_i = m_i g}$



Therefore $\boxed{\tau_i = d_i m_i g}$

$$\tau_i = -x_i m_i g$$

$$\tau_g = \sum_i \tau_i = -\sum_i x_i m_i g$$

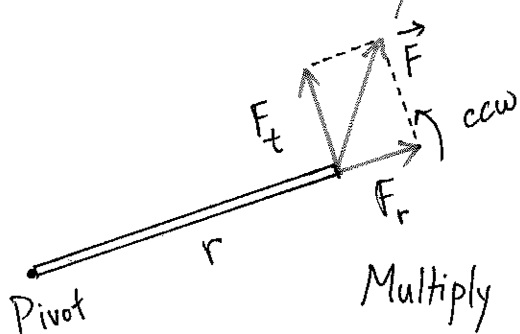
Remembering that $x_{CM} = \frac{1}{M} \sum_i m_i x_i$

$$\tau_g = -\underbrace{\left(\sum_i x_i m_i\right)}_{= x_{CM} \cdot M} g = -x_{CM} \underbrace{Mg}_{= F_G}$$

$= F_G$ for the whole object as a point particle.

$$\boxed{\tau_g = -x_{CM} M g}$$

12.6 Rotational Dynamics



Newton's 2nd Law in the t-coordinate:

$$F_t = m a_t = m r \alpha$$

Multiply both sides by r to get τ

$$r F_t = \boxed{\tau = m r^2 \alpha}$$

Analogous to Newton's 2nd Law.

For a Rigid Body,

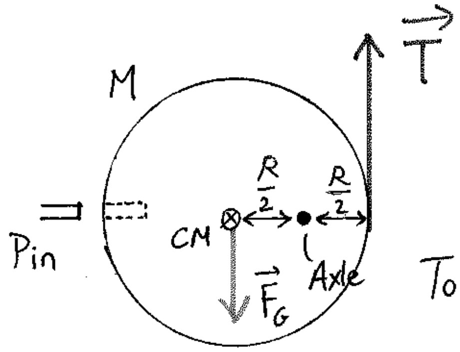
$$\tau_{net} = \sum_i \tau_i = \sum_i m_i r_i^2 \alpha$$

$$\tau_{net} = \underbrace{\left(\sum_i m_i r_i^2 \right)}_{= I} \alpha$$

$= I$ moment of inertia of the object.

$$\boxed{\tau_{net} = I \alpha}$$ Newton's 2nd Law for Rotational Motion.

12.7 Rotation about a fixed Axis



When the pin is removed, what is the initial angular acceleration of the disc?

$$\tau_{net} = \tau_g + \tau_{cable}$$

Torque due to gravity:

$$\tau_g = -Mg\,x_{CM} \quad w/ \quad x_{CM} = -\frac{1}{2}R$$

$$\tau_g = \frac{1}{2}MgR$$

Torque due to Tension: $\tau_{cable} = T \cdot \frac{R}{2}$

$$\tau_{net} = \frac{1}{2}MgR + \frac{1}{2}RT$$

$$\tau_{net} = \frac{1}{2}R(Mg+T)$$

The moment of Inertia about the axle:

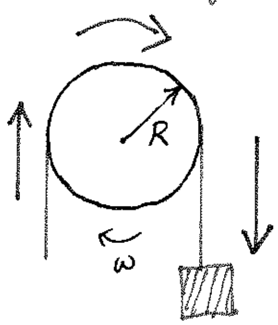
$$I = I_{CM} + M\left(\frac{R}{2}\right)^2 \quad , \quad I_{CM} = \frac{1}{2}MR^2$$

$$I = \frac{1}{2}MR^2 + \frac{1}{4}MR^2$$

$$I = \frac{3}{4}MR^2.$$

Finally, $\alpha = \dfrac{\tau_{net}}{I} = \dfrac{\frac{1}{2}R(Mg+T)}{\frac{3}{4}MR^2}$

Ropes and Pulleys

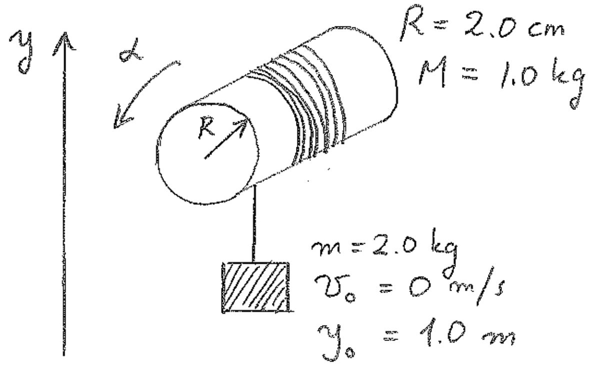


If the rope does not slip:

$$a_{block} = |\alpha| R$$

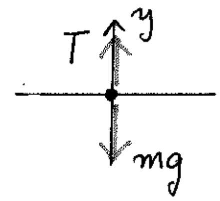
$$v_{block} = |\omega| R$$

Example



How long does it take for the block to reach the ground?

FBD of the block



$$\sum F_y = T - mg = ma_y$$

$$a_y = \frac{T - mg}{m}$$

If I can calculate a_y,

$$\Delta y = \frac{1}{2} a_y (\Delta t)^2$$

I can calculate Δt.

For the cylinder $\rightarrow$ $\alpha = \frac{\tau_{net}}{I} = \frac{TR}{\frac{1}{2}MR^2} = \frac{2T}{MR}$

Acceleration constraint $\rightarrow$ $a_y = -\alpha R$

$$a_y = -\frac{2T}{MR} R = -\frac{2T}{M} \rightarrow T = -a_y M/2$$

Substituting T to solve for a_y,

$$T - mg = ma_y$$

$$-\frac{1}{2} a_y M - mg = ma_y \rightarrow a_y\left(m + \frac{M}{2}\right) = -mg$$

$$a_y = \frac{-mg}{m + \frac{M}{2}}$$

The rest of the problem is kinematics:

$$\Delta y = \frac{1}{2} a_y (\Delta t)^2.$$