CHAPTER 12

* Particle model vs. Rigid Body model
* Basic types of motion for a rigid body
  1) Translational Motion
  2) Rotational Motion
  3) Combination of 1 + 2.

Rotational Kinematics.

Angular velocity : $\omega = \dfrac{d\theta}{dt}$ ($\omega > 0$ for ccw motion)

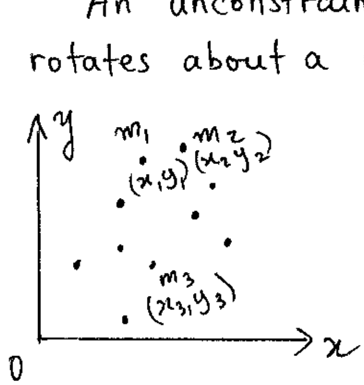
Angular acceleration : $\alpha = \dfrac{d\omega}{dt}$

$\omega > 0$	$\omega > 0$	$\omega < 0$	$\omega < 0$
$\alpha > 0$	$\alpha < 0$	$\alpha > 0$	$\alpha < 0$
speeding up ccw	slowing down ccw	slowing down cw	speeding up cw

$$\boxed{\begin{array}{ll} v_r = 0 & a_r = \dfrac{v^2}{r} = \omega^2 r \\ v_t = r\omega & a_t = r\alpha \end{array}}$$

12.2 Rotation about the center of mass

An unconstrained object on which there is no net Force, rotates about a point called "center of mass".



$$\begin{cases} x_{cm} = \dfrac{m_1 x_1 + m_2 x_2 + \cdots}{m_1 + m_2 + \cdots} = \dfrac{1}{M} \sum_i m_i x_i \\ y_{cm} = \dfrac{m_1 y_1 + m_2 y_2 + \cdots}{m_1 + m_2 + \cdots} = \dfrac{1}{M} \sum_i m_i y_i \end{cases}$$

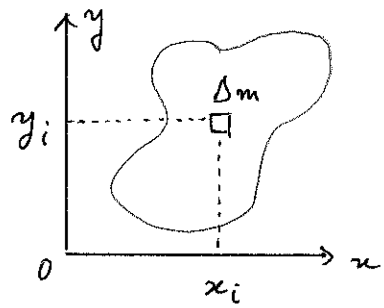
(x_{cm}, y_{cm}) are the center of mass coordinates.

* When $m_1 = m_2 = m_3 = \cdots = m$ for N particles,

$$x_{cm} = \frac{m(x_1 + x_2 + \cdots)}{Nm} = \frac{x_1 + x_2 + \cdots}{N} \rightarrow \text{Average } x\text{-coordinate}$$

$$y_{cm} = \frac{m(y_1 + y_2 + \cdots)}{Nm} = \frac{y_1 + y_2 + \cdots}{N} \rightarrow \text{Average } y\text{-coordinate}$$

* For a Rigid Body of mass M,



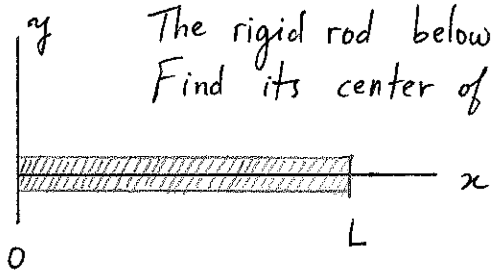
$$x_{cm} = \frac{1}{M} \sum_i^N x_i \, \Delta m \quad \underline{\text{and}} \quad y_{cm} = \frac{1}{M} \sum_i^N y_i \, \Delta m$$

As $\Delta m \rightarrow 0$ and $N \rightarrow \infty$

$$\boxed{x_{cm} = \frac{1}{m} \int x \, dm \quad \underline{\text{and}} \quad y_{cm} = \frac{1}{M} \int y \, dm}$$

<u>Example</u>

The rigid rod below is of length L and mass M. Find its center of mass.



The density per length: $\lambda = \frac{M}{L}$

$dm = \lambda \, dx = \frac{M}{L} dx$

$$x_{cm} = \frac{1}{M} \int x \, dm = \frac{1}{M} \int_0^L x \frac{M}{L} dx$$

$$x_{cm} = \frac{1}{L} \int_0^L x \, dx = \frac{1}{L} \left[\frac{x^2}{2} \right]_0^L$$

$\boxed{x_{cm} = \frac{L}{2}}$ as expected.

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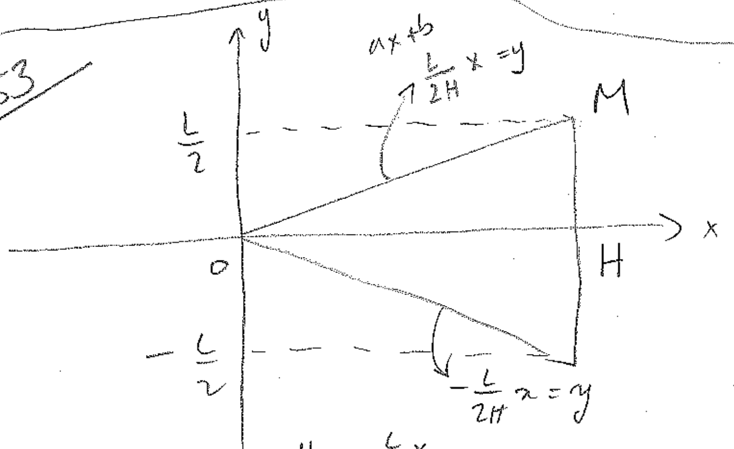


$$x_{cm} = \frac{1}{M}\int x \, dm$$

$$dm = \sigma \, dA = \frac{M}{A} dx\,dy$$

$$dm = \frac{2M}{HL} dx\,dy$$

$$y_{cm} = \frac{1}{M}\int_0^H \left(\int_{-\frac{L}{2H}x}^{+\frac{L}{2H}x} y \, \frac{2M}{HL} dy \right) dx$$

$$y_{cm} = \frac{2M}{MHL2}\int_0^H 0 \, dy = 0$$

For $L = 20$ cm and $H = 30$ cm

$$\begin{cases} x_{cm} = \frac{2}{3}\, 30\text{ cm} = 20\text{ cm} \\ y_{cm} = 0\text{ cm} \end{cases}$$

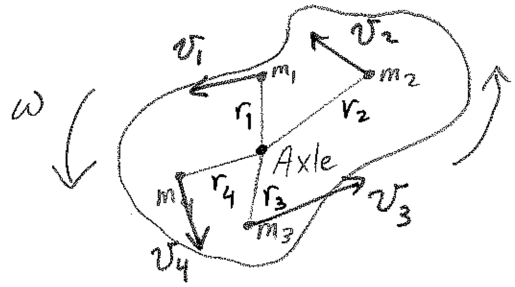
$$x_{cm} = \int_{-\frac{L}{2H}x}^{\frac{L}{2H}x} \underbrace{\int_0^H x \frac{2M}{HL} dx}\, dy = \int_0^H \left(\int_{-\frac{L}{2H}x}^{\frac{L}{2H}x} dy \right) \frac{2M}{HL} x \, dx$$

$$x_{cm} = \frac{1}{M}\frac{2M}{H\cancel{L}}\int_0^H \frac{\cancel{L}}{H} x \, x \, dx = \frac{1}{M}\frac{2M}{H^2 3} x^3 \Big|_0^H = \frac{2H}{3} = \frac{2}{3}H$$

12.3 Rotational Energy

The kinetic energy due to rotation is called rotational kinetic energy.

For a Rigid object:



$$K_{rot} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \dots$$

$$K_{rot} = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \dots$$

$$= \frac{1}{2} \underbrace{\left(\sum_i m_i r_i^2 \right)}_{\equiv I \to \text{Moment of Inertia}} \omega^2$$

Remember that $v = r\omega$

$$\boxed{K_{rot} = \frac{1}{2} I \omega^2}$$

* Note that $\frac{1}{2} I \omega^2$ is analogous to $\frac{1}{2} m v^2$.
* I depends on the axis of rotation.
* I depends on how the mass of the object is distributed.
* If the rotation axis (axle) is not through the center of mass of the object

$$\boxed{E_{mech} = K_{rot} + U_g = \frac{1}{2} I \omega^2 + M g y_{cm}}$$

* For a continuous distribution of mass, i.e. a rigid body

$$\boxed{I = \int r^2 \, dm}$$

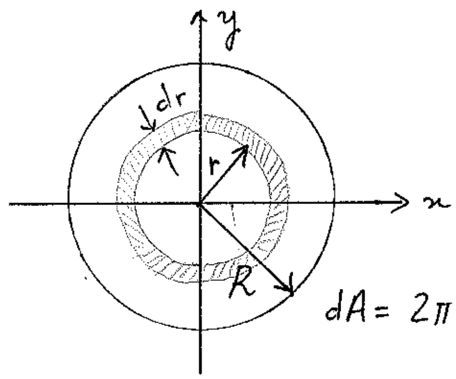
Example Calculate the moment of inertia of a uniform rigid rod of mass M and length L that rotates about a pivot at one end.

Density $\lambda = \frac{M}{L} \to dm = \frac{M}{L} dx$

$$I = \int x^2\, dm = \int_0^L x^2 \frac{M}{L}\, dx = \frac{M}{L}\int_0^L x^2\, dx$$

$$I = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \boxed{\frac{1}{3}ML^2}$$

Example
Find the moment of inertia for a circular disc of radius R and mass m that rotates on an axis passing through its center.



$dA = 2\pi r\, dr$

Density per unit area $\sigma = \dfrac{M}{\pi R^2}$

$$dm = \sigma\, dA = \frac{M}{\pi R^2}\, 2\pi r\, dr$$

$$I_{disc} = \int r^2\, dm$$

$$I_{disc} = \int_0^R r^2 \frac{M}{\pi R^2}\, 2\pi r\, dr$$

$$I_{disc} = \frac{2M}{R^2}\int_0^R r^3\, dr$$

$$= \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \boxed{\frac{1}{2}MR^2}$$

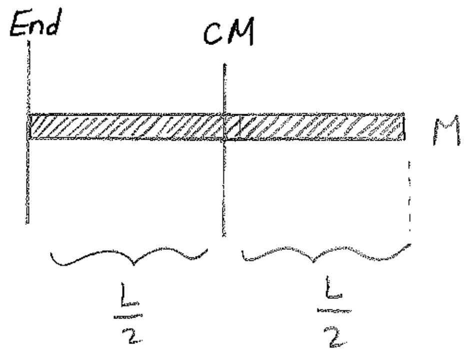
The parallel axis theorem

If the axis of interest is a distance d from a parallel axis through the center of mass, the moment of inertia

$$\boxed{I = I_{cm} + Md^2}$$

Example
Calculate the moment of inertia of a uniform rigid rod of mass M and Length L that rotates about a pivot through its center of mass?

We have already calculated I for a rod rotating about a pivot at one end.



$$I_{end} = \frac{1}{3} M L^2$$

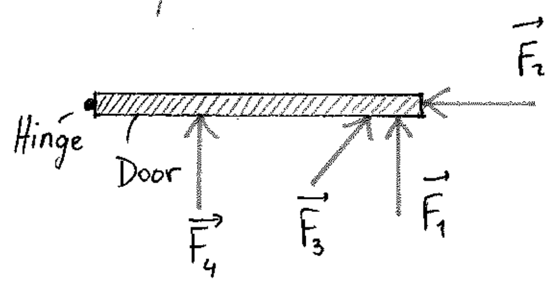
$$I_{end} = I_{CM} + M d^2$$

$$I_{end} = I_{CM} + M \left(\frac{L}{2}\right)^2$$

$$\frac{1}{3} M L^2 = I_{CM} + \frac{1}{4} M L^2$$

$$I_{CM} = \left(\frac{1}{3} - \frac{1}{4}\right) M L^2 = \boxed{\frac{1}{12} M L^2}$$

12.5 Torque



All $\vec{F}_1$, $\vec{F}_3$ and $\vec{F}_4$ will open the door. $\vec{F}_2$ will not.

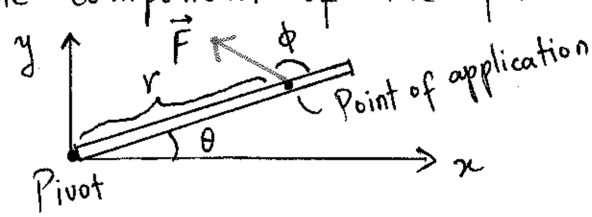
$\vec{F}_1$ will open it the most easily.

$\vec{F}_4$ opens it but not as easy as $\vec{F}_1$.

The ability of a force to cause rotation depens on:

1) Its magnitude.
2) The distance r from the point of application to the pivot point.
3) The angle.

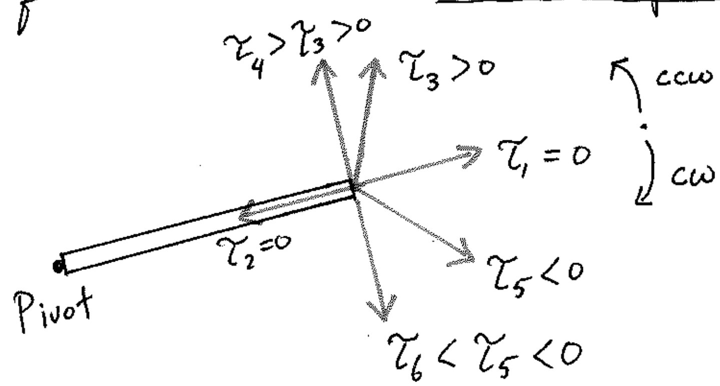
Combining 1 & 3 we can say that it depens on the magnitude of the component of the force in the direction of rotation.



Torque is defined as

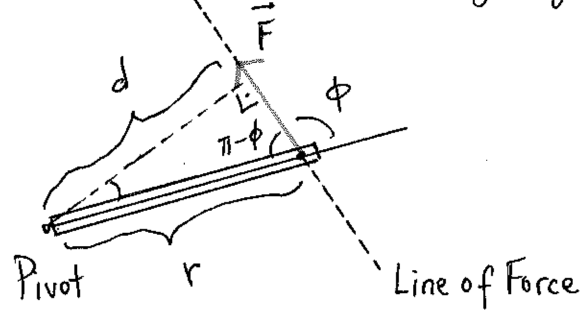
$$\boxed{\tau = r F \sin\phi}$$

* τ is the rotational equivalent of Force.
* Units of τ is $N \cdot m$ in SI units.
* τ is a vector like F. We'll use its sign to denote direction:
  1) $\tau > 0$ for torque causing ccw rotation
  2) $\tau < 0$ for torque causing cw rotation
* Torque is calculated <u>about a pivot point.</u>



Alternatively, $\boxed{\tau = r F_t}$ where F_t is the component of force perpendicular to the radial line. $\boxed{F_t = F \sin\phi.}$

Yet another alternative way of looking at τ;



$$d = r \sin(\pi - \phi) = r \sin\phi$$

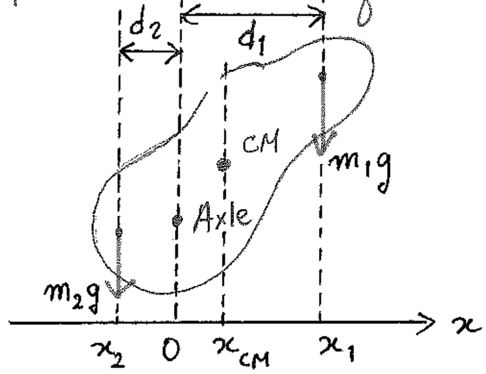
$$\boxed{|\tau| = d F = F r \sin\phi}$$

d is the distance from the pivot point to the line of $\vec{F}$.

* d is called "moment arm".

Net torque : $\tau_{net} = \tau_1 + \tau_2 + \cdots = \sum_i \tau_i$

Gravitational Torque:



Gravitational Force acting on every particle making up the object is $\boxed{F_i = m_i g}$

Therefore $\boxed{\tau_i = d_i m_i g}$

$$\tau_i = -x_i m_i g$$

$$\tau_g = \sum_i \tau_i = -\sum_i x_i m_i g$$

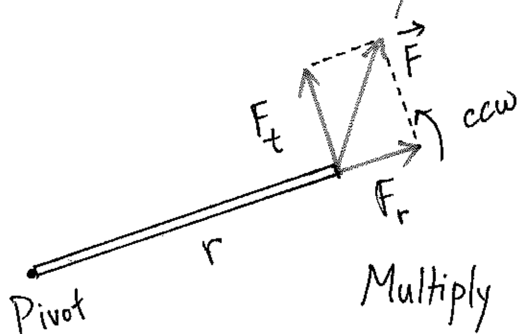
Remembering that $x_{CM} = \frac{1}{M} \sum_i m_i x_i$

$$\tau_g = -\underbrace{\left(\sum_i x_i m_i\right)}_{= x_{CM} \cdot M} g = -x_{CM} \underbrace{Mg}_{= F_G}$$

$= F_G$ for the whole object as a point particle.

$$\boxed{\tau_g = -x_{CM} Mg}$$

12.6 Rotational Dynamics



Newton's 2nd Law in the t-coordinate:

$$F_t = m a_t = m r \alpha$$

Multiply both sides by r to get τ

$$r F_t = \boxed{\tau = m r^2 \alpha}$$

Analogous to Newton's 2nd Law.

For a Rigid Body,

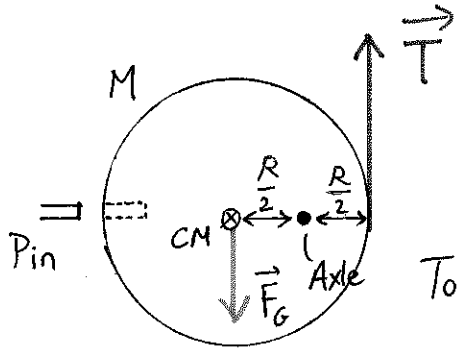
$$\tau_{net} = \sum_i \tau_i = \sum_i m_i r_i^2 \alpha$$

$$\tau_{net} = \underbrace{\left(\sum_i m_i r_i^2 \right)}_{= I} \alpha$$

$= I$ moment of inertia of the object.

$$\boxed{\tau_{net} = I \alpha}$$ Newton's 2nd Law for Rotational Motion.

12.7 Rotation about a fixed Axis



When the pin is removed, what is the initial angular acceleration of the disc?

$$\tau_{net} = \tau_g + \tau_{cable}$$

Torque due to gravity:

$$\tau_g = -Mg\,x_{CM} \quad w/ \quad x_{CM} = -\frac{1}{2}R$$

$$\tau_g = \frac{1}{2}MgR$$

Torque due to Tension: $\tau_{cable} = T \cdot \frac{R}{2}$

$$\tau_{net} = \frac{1}{2}MgR + \frac{1}{2}RT$$

$$\tau_{net} = \frac{1}{2}R(Mg + T)$$

The moment of Inertia about the axle:

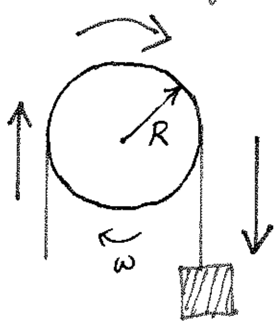
$$I = I_{CM} + M\left(\frac{R}{2}\right)^2 \quad , \quad I_{CM} = \frac{1}{2}MR^2$$

$$I = \frac{1}{2}MR^2 + \frac{1}{4}MR^2$$

$$I = \frac{3}{4}MR^2.$$

Finally, $\alpha = \dfrac{\tau_{net}}{I} = \dfrac{\frac{1}{2}R(Mg+T)}{\frac{3}{4}MR^2}$

Ropes and Pulleys

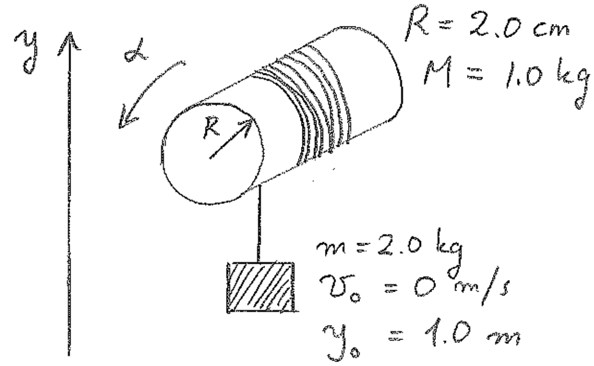


If the rope does not slip:

$$a_{block} = |\alpha| R$$

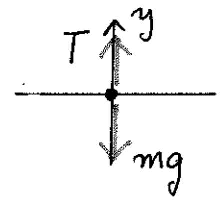
$$v_{block} = |\omega| R$$

Example



How long does it take for the block to reach the ground?

FBD of the block



$$\sum F_y = T - mg = ma_y$$

$$a_y = \frac{T - mg}{m}$$

If I can calculate a_y,

$$\Delta y = \frac{1}{2} a_y (\Delta t)^2$$

I can calculate Δt.

For the cylinder $\rightarrow$ $\alpha = \frac{\tau_{net}}{I} = \frac{TR}{\frac{1}{2}MR^2} = \frac{2T}{MR}$

Acceleration constraint $\rightarrow$ $a_y = -\alpha R$

$$a_y = -\frac{2T}{MR} R = -\frac{2T}{M} \rightarrow T = -a_y M/2$$

Substituting T to solve for a_y,

$$T - mg = ma_y$$

$$-\frac{1}{2} a_y M - mg = ma_y \rightarrow a_y\left(m + \frac{M}{2}\right) = -mg$$

$$a_y = \frac{-mg}{m + \frac{M}{2}}$$

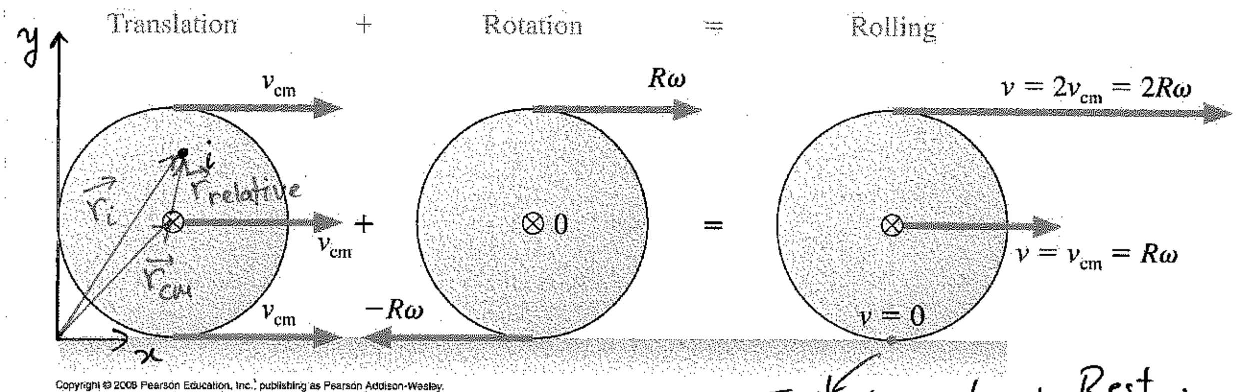
The rest of the problem is kinematics:

$$\Delta y = \frac{1}{2} a_y (\Delta t)^2.$$

12.8 Static Equilibrium $\rightarrow$ $\boxed{\tau_{net} = 0.}$

12.9 Rolling Motion

* Rolling is combination of translation of CM and rotation.



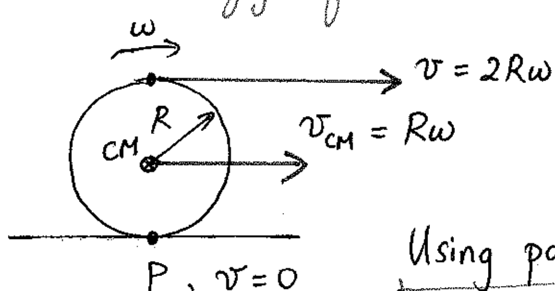
Instantaneously at Rest.

The velocity of point i $\rightarrow$ $\vec{v}_i = \vec{v}_{CM} + \vec{v}_{\text{Relative to CM}}$

This follows from $\rightarrow$ $\vec{r}_i = \vec{r}_{CM} + \vec{r}_{\text{Relative to CM}}$

Taking time derivative $\frac{d\vec{r}_i}{dt} = \vec{v}_i$.

Kinetic Energy of a Rolling Object:



$$\boxed{K = K_{\text{rotation about P}} = \frac{1}{2} I_P \omega^2}$$

Using parallel axis theorem

$$\boxed{I_P = I_{CM} + MR^2}$$

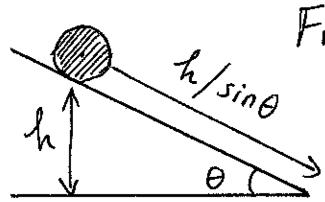
Therefore,

$$K = \frac{1}{2}\left(I_{CM} + MR^2\right)\omega^2 = \underbrace{\frac{1}{2} I_{CM}\,\omega^2}_{K_{rot}} + \underbrace{\frac{1}{2} M (R\omega)^2}_{K_{trans}}$$

$$K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v^2$$
$$K = K_{rot} + K_{trans}$$

The Downhill Race

Consider a rolling object on an inclined plane, a distance h from the ground.



From Conservation of Energy

$$\boxed{\frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v_{CM}^2 = Mgh}$$

For all the objects we consider, $I_{CM} = cMR^2$ where c is a constant and depends on the shape of the object.

$$\frac{1}{2} cMR^2 \left(\frac{v_{CM}}{R}\right)^2 + \frac{1}{2} M v_{CM}^2 = Mgh$$

$$\frac{1}{2} M (c+1) v_{CM}^2 = Mgh$$

$$\boxed{v_{CM} = \left(\frac{2gh}{c+1}\right)^{1/2}}$$

To obtain the acceleration of CM; $v_{CM}^2 = 2a_{CM} \Delta x$

$$\frac{2gh}{c+1} = 2a_{CM} \Delta x = 2a_{CM} \frac{h}{\sin\theta}$$

$$\boxed{a_{CM} = \frac{g \sin\theta}{c+1}}$$

For a point particle moving down the inclined plane

$$a_{particle} = g \sin\theta \quad \text{(From FBD and Newton's 2nd Law)}$$

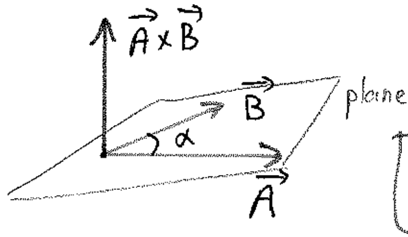
Therefore, $\boxed{a_{CM} = \dfrac{a_{particle}}{c+1}}$

i.e. $\rightarrow$ $\boxed{a_{CM} < a_{particle}}$ $\rightarrow$ Some of the total energy goes into rolling motion.

12.10 The Vector Description of Rotational Motion

* The cross product of two vectors $\vec{A}$ and $\vec{B}$

$$\vec{A} \times \vec{B} = (AB \sin\alpha, \text{ in the direction given by the right hand rule})$$



* $\vec{A}\times\vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$

$\boxed{* \ |\vec{A}\times\vec{B}| = AB \sin\alpha}$

* $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

* In rectangular coordinates: $\boxed{\hat{i}\times\hat{j} = \hat{k} \ ; \ \hat{k}\times\hat{i} = \hat{j} \ ; \ \hat{j}\times\hat{k} = \hat{i}}$

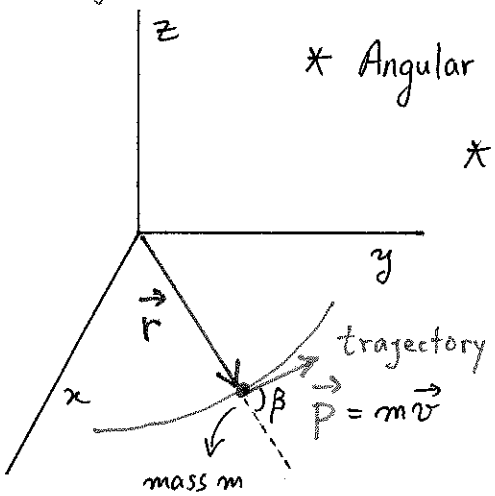
* Derivatives are associative

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

Torque: $\boxed{\vec{\tau} = \vec{r}\times\vec{F}}$ $\rightarrow$ $\tau = rF\sin\alpha$

* Torque is perpendicular to the plane of $\vec{r}$ and $\vec{F}$.

Angular Momentum



* Angular momentum $\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$
* $\boxed{|\vec{L}| = rp\sin\beta = mrv\sin\beta}$
* $\vec{L}$ is perpendicular to the plane of motion.
* Units of $\vec{L}$ is $\frac{kg\,m^2}{s}$.
* For circular motion $\beta = 90°$

$\boxed{L_z = mrv_t.}$ Positive for ccw motion. $\boxed{L_z = mr^2\omega}$

* Newton's 2nd Law : ($\vec{F}_{net} = d\vec{p}/dt$)

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r}\times\vec{p})$$

$$= \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt}$$

$$= \underbrace{\vec{v}\times\vec{p}}_{=0} + \vec{r}\times\vec{F}_{net} \longrightarrow \boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_{net}}$$

12.11 Angular Momentum of a Rigid Body

Imagining a rigid body as consisting of many point particles,

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots = \sum_i \vec{L}_i$$

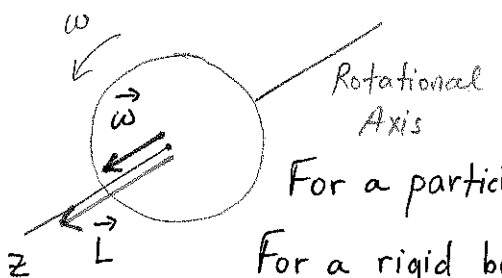
Taking time derivative of both sides, we get

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{L}_i}{dt} = \sum_i \vec{\tau}_i = \vec{\tau}_{net}$$

Conservation of Angular Momentum

The angular momentum $\vec{L}$ of an isolated system ($\vec{\tau}_{net} = 0$) is conserved, i.e. $\boxed{\vec{L}_i = \vec{L}_f}$

Angular Momentum & Angular velocity :



$$\vec{L} = I\vec{\omega}$$

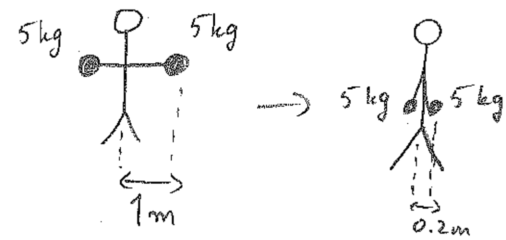
For a particle $L_z = m r v_t = m r^2 \omega \quad (v_t = r\omega)$

For a rigid body : $L_z = \left(\sum_i m_i r_i^2\right)\omega = I\omega.$

Example 1. A man stands at the center of a turntable holding his arms extended horizontally with a 5 kg mass in each hand. He is set in motion with an angular velocity ω = 5 rev/s. Assume that the moment of inertia of a man is about 6 kg·m^2 and that his arms are 1 meter in length. What is his angular speed if he drops his arms to his sides resulting a final distance of the masses from his center of rotation of 0.2 meters?

$I_{man} = 6 \ kg\,m^2 \quad , \quad \omega = 5 \ \frac{rev}{s}$

Compute the total moment of inertia:



$$I_{total} = I_{man} + I_{weights} = 6kg \cdot m^2 + mr^2$$

$$\rightarrow I_{initial} = \underbrace{6kg \cdot m^2}_{I_{man}} + \underbrace{10kg \cdot m^2}_{(5kg+5kg)\,1^2 m^2} = 16kg \cdot m^2$$

$$\rightarrow I_{final} = \underbrace{6kg \cdot m^2}_{I_{man}} + \underbrace{0.4kg \cdot m^2}_{(5kg+5kg)\,(0.2)^2\,m^2} = 6.4kg \cdot m^2$$

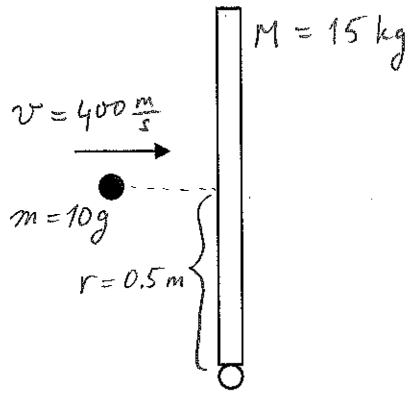
$I_i \omega_i = I_f \omega_f$

Now conserve angular momentum:

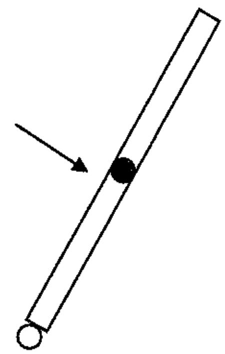
$$I_i\omega_i = I_f\omega_f \rightarrow \frac{I_i\omega_i}{I_f} = \omega_f \rightarrow \frac{(16kg \cdot m^2)(5rev \cdot s^{-1})}{6.4kg \cdot m^2} = 12.5rev \cdot s^{-1}$$

Example 3. A bullet, mass = 10 grams, is fired into the center of a door, mass = 15 kg, 1 meter wide with a velocity of 400 m/s. The door is mounted on frictionless hinges. Find the angular speed of the door after the impact.

Consider the type of collision involved here. There is no net external torque exerted on the bullet-door system so angular momentum *is* conserved. The bullet does exert a torque on the door but the door, in return exerts a torque on the bullet so the condition of zero external torques is met.



system before collision



system after collision

Computing angular momentum with respect to the door hinge:

Before: $\ell_{bullet} = mvr = (0.01kg)(400m \cdot s^{-1})(0.5m) = 2.0kg \cdot m^2 \cdot s^{-1}$

$\ell_{door} = 0$

After: $\ell_{system} = I_{system}\omega$

$I_{sys} = I_{door} + I_{bullet}$

$$I_{door} = \frac{ML^2}{3} = \frac{(15kg)(1.0m)^2}{3} = 5kg \cdot m^2$$

$$I_{bullet} = MR^2 = (0.01kg)(0.5m)^2 = 0.0025kg \cdot m^2$$

$\because$ conservation of angular momentum requires:

$$mvr = I_{system}\omega$$

$$2.0kg \cdot m^2 \cdot s^{-1} = (5.0025kg \cdot m^2)\omega$$

$$\omega = 0.4rad \cdot s^{-1}$$

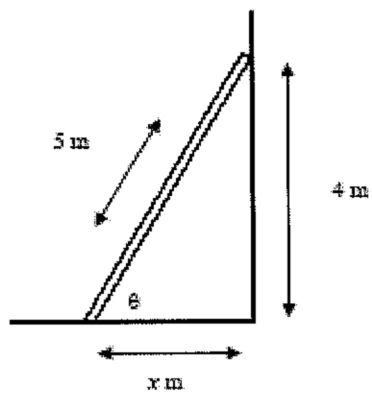
Is energy conserved? $KE_i = \frac{1}{2}m_{sys}v^2 = 800J$, $KE_f = \frac{1}{2}I\omega^2 = 0.4J$

1/2000 of the initial value!

Example 3. A 5 meter long ladder leans against a frictionless wall. The point of contact between the ladder and the wall is 4 meters above the ground. The ladder is uniform with a mass of 12 kg. Determine the forces exerted by the ground and wall on the ladder.

$\Sigma F = 0$

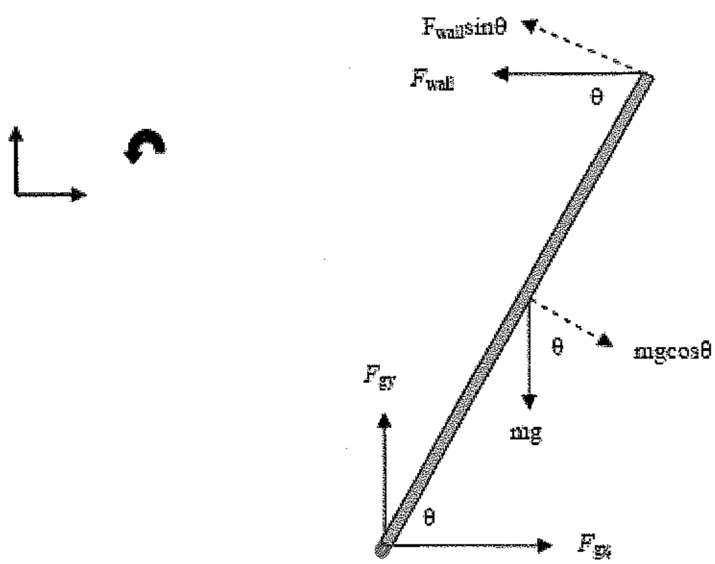
$\Sigma \tau = 0$



Wall is frictionless.

A little yields $\theta = 53°$ and $x = 3\text{m}$.

FBD



Notice that since the wall is frictionless the force that it exerts on the latter is normal to the surface of the wall. It is necessary to find the component perpendicular to the ladder only for the purpose of computing a torque. The force that the ground exerts on the ladder, however, does have two components. Can you explain why?

Applying the **first condition** yields:

$$\sum F_y = F_{gy} - mg = 0$$

$$F_{gy} = mg \therefore F_{gy} = 118N$$

$$\sum F_x = F_{gx} - F_{wall} = 0$$

$$F_{gx} = F_{wall}$$

Applying the second condition with respect to the point of contact between the ground and latter (this eliminates F_g and it's components from torque computations):

$$\sum \Gamma = (F_{wall})(\sin\theta)(5m) - (mg)(\cos\theta)(2.5m) = 0$$

$$\sum \Gamma = (F_{wall})(4m) - (71N)(2.5m)$$

$$F_{wall} = 44N$$

Recall that $F_{gx} = F_{wall} \therefore F_{gx} = 44N$ and:

$$F_g = \sqrt{(44N)^2 + (118N)^2} = 126N$$

$$\tan\phi = \frac{y}{x} = \frac{118N}{44N} \therefore \theta = 70°$$

$$\vec{F}_g = 126N \text{ @ } 70°$$