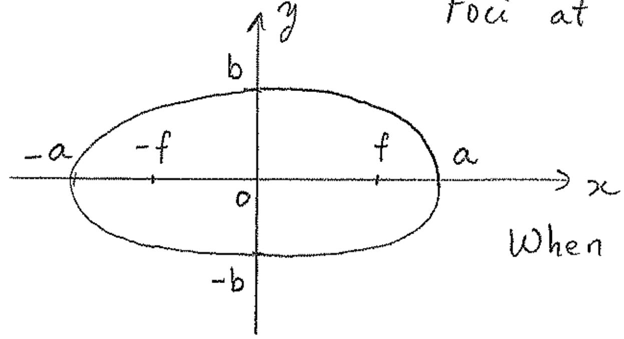
# CHAPTER 13

## Ellipses




Foci at $\pm f$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

When $a = b$ we get a circle.

Focus $\rightarrow$ $f = \sqrt{a^2 - b^2}$

Eccentricity $\rightarrow$ $\epsilon = \frac{\sqrt{a^2-b^2}}{a} = \frac{f}{a}$

$a$ is semi-major axis length

$b$ is semi-minor axis length

Area $\rightarrow$ $\pi a b$

## Reasoning behind Newton's Law of Gravity

* Centripetal acceleration of an object in circular motion

$$a_r = \frac{v^2}{r} \rightarrow \text{For any object on earth.}$$

Assuming same laws also apply to moon,

$$a_{moon} = g_{at\ moon} = \frac{v_m^2}{r_m}$$

Here $v_m = 2\pi r_m / T_m$ $\rightarrow$ 27.3 days

$$\boxed{g_{at\ moon} = \frac{4\pi^2 r_m}{T_m^2}}$$

* From kinematics experiments with dropped objects on earth;

$$g_{\text{on earth}} = 9.80 \ m/s^2 .$$

* Then, Newton looked at the ratio $g_{\text{at moon}} / g_{\text{on earth}}$

$$\frac{g_{\text{at moon}}}{g_{\text{on earth}}} \simeq \frac{1}{3600}$$

* He also realized that $\dfrac{r_m}{R_{\text{earth}}} \simeq 60$ . and $60^2 = 3600$.

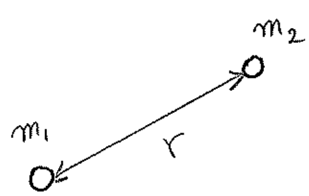
$$\frac{g_{\text{at moon}}}{g_{\text{on earth}}} = \left( \frac{R_{\text{earth}}}{r_m} \right)^2$$

OR $$\boxed{\frac{g_{\text{at moon}}}{g_{\text{on earth}}} = \left( \frac{1/r_m}{1/R_{\text{earth}}} \right)^2}$$ → Inverse square Law!.

* The key insight here was that same laws of physics govern the motion of planets, moon, etc. as those on earth. That is, **laws of physics are universal.**

## Newton's Law of Gravity

$$F_{1\,\text{on}\,2} = F_{2\,\text{on}\,1} = \frac{G \, m_1 m_2}{r^2}$$



# Principle of Equivalence

* Inertial mass of an object:

$$\boxed{m_{inertial} = \frac{F}{a}} \rightarrow \text{This has } \underline{\text{nothing}} \text{ to do w/ gravity.}$$

* Gravitational mass of an object:

$$\boxed{m_{grav} = \frac{r^2 F_{M\,on\,m}}{GM}} \qquad \text{from} \quad F_{M\,on\,m} = \frac{GMm}{r^2}$$

* $\boxed{m_{inertial} = m_{grav}}$ is called principle of equivalence.

# Newton's <u>Theory</u> of Gravity

1) Newton's Law of Gravity : $F_{m\,on\,M} = \dfrac{GMm}{r^2}$
2) The principle of Equivalence.
3) Laws of motion are <u>universal</u>.

# Little $\boxed{g}$ and Big $\boxed{G}$

* On planet X $\rightarrow$ $F_{G\,on\,x} = m\,g_{on\,x}$
* Also $\rightarrow$ $F_{M_x\,on\,m} = \dfrac{G M_x m}{R^2}$

$$\boxed{g_{on\,x} = \frac{GM_x}{R^2}}$$

* Note that g decreases with distance as $1/r^2$.

Example

A planet has 4 times the mass of the earth, but the acceleration due to gravity on the planet's surface is the same as on the earth's surface. What is the planet's radius?

$$g_{earth} = \frac{G M_{earth}}{R_e^2} \quad \text{and} \quad g_X = \frac{G M_X}{R_X^2} \; ; \; M_X = 4 M_{earth}$$

$$g_{earth} = g_X$$

$$\frac{\cancel{G}\cancel{M_{earth}}}{R_e^2} = \frac{4\cancel{G}\cancel{M_{earth}}}{R_X^2}$$

$$R_X^2 = 4R_e^2 \rightarrow \boxed{R_X = 2R_e}$$

Gravitational Potential Energy.

$$\Delta U = -W_c(i \rightarrow f) = -\int_{x_i}^{x_f} F_x \, dx$$

* Before, we used

$$F = -mg \quad \text{and choose} \quad U = 0 \;@\; y = 0.$$

to get $U_g = mgy$. → This is true in the flat earth approximation when $g$ is constant.

$$\Delta U = \underbrace{U_{at\,\infty}}_{\text{Choose to be 0.}} - U_{at\,r} = -\int_r^{\infty} (F_{1on2})_x \, dx$$

$$-U_{\text{at } r} = -\int_r^{\infty} -\frac{GMm}{x^2}\,dx = GMm\int_r^{\infty}\frac{dx}{x^2}$$

$$-U_{\text{at } r} = -GMm\,\frac{1}{x}\Big|_r^{\infty}$$

$$U_{\text{at } r} \equiv \boxed{U_g = -\frac{GMm}{r}}$$

We integrated F along the x-axis but since F is conservative, this holds for <u>any path.</u>

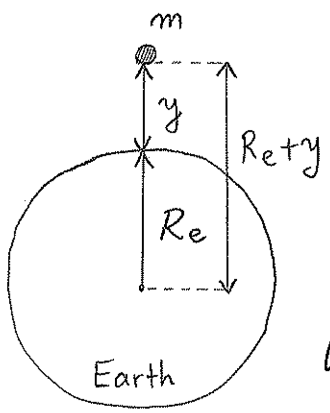
<u>Example</u> What is the "escape speed" for a 1000 kg rocket from the earth?

Energy Conservation: $\underbrace{\cancelto{0}{K_f} + \cancelto{0}{U_{g_f}}}_{@\ r\to\infty} = K_i + U_{g_i}$

$$0 + 0 = \frac{1}{2}\cancel{m}v_i^2 - \frac{GM_e\cancel{m}}{R_e}$$

$$v_i = \left(\frac{2GM_e}{R_e}\right)^{1/2} \approx 25{,}000 \text{ mph.}$$

# The Flat Earth Approximation




$$U_g = -\frac{GM_e m}{r} = -\frac{GM_e m}{R_e + y}$$

$$U_g = -\frac{GM_e m}{R_e(1 + y/R_e)}$$

Using the binomial approximation:
$(1+x)^n \approx 1 + nx$ if $x \ll 1$.

$$U_g \simeq -GM_e m \frac{1}{R_e}(1 - y/R_e)$$

$$U_g \simeq \underbrace{-\frac{GM_e m}{R_e}}_{\substack{\text{Gravitational} \\ \text{Potential @ surface} \\ \text{of earth.} \equiv U_0}} + m\underbrace{\left[\frac{GM_e}{R_e^2}\right]}_{= g_e} y \rightarrow \boxed{U_g = U_0 + mg_e y \text{ if } y \ll R_e}$$

Flat - Earth Approx.

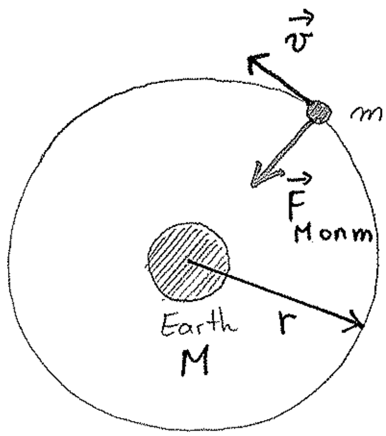
* If we redefine our reference point for $U_g$ to 0 rather than $U_0$,

$$\boxed{U_g = mg_e y}$$ Flat - Earth approximation.

## 13.6 Satellite Orbits and Energies

Our analysis will be for circular orbits.

For earth, $\frac{\text{semi-minor axis}}{\text{semi-major axis}} \simeq 0.99986 \rightarrow$ very close to being <u>circular.</u>




$$F_{M on m} = \frac{GMm}{r^2} = \frac{mv^2}{r} = ma_r$$

Then speed of a satellite is

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\boxed{v = \sqrt{\frac{GM}{r}}} \longrightarrow \text{Independent of } m\,!$$

* m is in free fall towards the earth

Example

The space shuttle in a 300 km high orbit ($\approx 180$ mi) wants to capture a smaller satellite for repair. What are the speeds of the shuttle and the satellite in the orbit?

$$v_{shuttle} = v_{satellite} = v = \sqrt{\frac{(6.67\times10^{-11}\ Nm^2/kg^2)(5.98\times10^{24}\ kg)}{\underbrace{3.0\times10^5\ m}_{h} + \underbrace{6.37\times10^6\ m}_{R_e}}}$$

$$v = 7730\ m/s \approx 17{,}000\ mph.$$

Kepler's Third Law

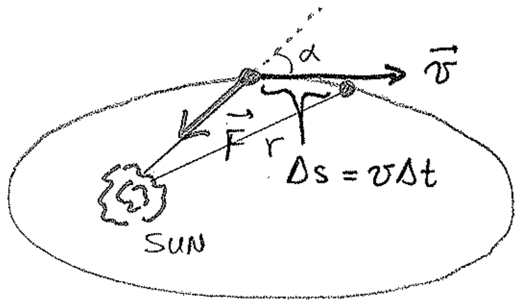
Period of the orbit is T $\longrightarrow$ $v = \frac{2\pi r}{T}$

For a circular orbit $\rightarrow$ $v = \sqrt{\frac{GM}{r}} = \frac{2\pi r}{T}$

$$\boxed{T^2 = \left(\frac{4\pi^2}{GM}\right) r^3}$$

* Square of the period is proportional to the cube of the radius, This is <u>Kepler's 3rd Law.</u>

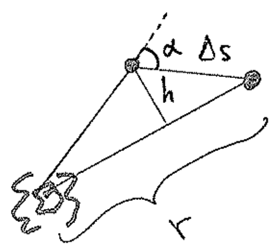
## Kepler's 2nd Law




In small time interval $\Delta t$, the area swept is,

$$\Delta A = \frac{1}{2} \times \text{base} \times \text{height}$$

$$\Delta A = \frac{1}{2} r \underbrace{\Delta s}_{= v\Delta t} \sin\alpha$$




$h = \Delta s \sin\alpha$

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r v \sin\alpha \cdot \frac{m}{m}$$

$$\frac{\Delta A}{\Delta t} = \frac{m v r \sin\alpha}{2m} = \frac{L}{2m}$$

The angular momentum $L$ is conserved since $\tau_{net} = 0$,

$$\boxed{\frac{\Delta A}{\Delta t} = \text{Constant}}$$ Kepler's 2nd Law

## Kinetic Energy vs. Potential Energy in Orbit

A satellite's KE $\rightarrow$ $K = \frac{1}{2} m v^2$

For a circular orbit $\rightarrow$ $v = \sqrt{\frac{GM}{r}}$

$\Rightarrow K = \frac{1}{2} m \frac{GM}{r}$

$$\boxed{K = \frac{1}{2} \underbrace{\frac{GMm}{r}}_{-U_g} = -\frac{1}{2} U_g}$$ For a circular orbit.

$$E_{mech} = K + U_g$$

$$E_{mech} = -\frac{1}{2}U_g + U_g = \frac{1}{2}U_g$$

* Note that $U_g < 0$ hence $E_{mech} < 0$. This is characteristic of a "bound" system.

* $E_{mech}$ for an unbounded system must be positive since $U \to 0$ as $r \to \infty$ and $K \geq 0$ always.